Introduction to Quantitative Methods, Quiz 5

1. Give the definition of the following statements:
   (a) (15 points) A set $E \subset M$ is an *open set*.
   (b) (15 points) A set $F \subset M$ is an *closed set*.
2. Answer the following questions.
   (a) (15 points) Prove that if E is open, then the set E^c, the complement of E, is closed.
   (b) (15 points) Suppose X is an open set and Y is a closed subset of X. Prove that $X \setminus Y$ is open.
3. (20 points) Suppose $S \subset \mathbb{R}$ such that S has an upper bounded. Prove that the supremum of S is a limit point or an element of S.
4. Answer the following questions.
   (a) (20 points) Prove that the intersection of finite open sets is open.
   (b) (10 points) Consider the intersection of countably many open intervals defined as follows:

$$E = \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n} \, , \ \frac{1}{n} \right).$$

   Is E an open set? Justify your answer.

1. Give the definition of the following statements:

   (a) (15 points) A set $E \subset M$ is an *open set.*

   (b) (15 points) A set $F \subset M$ is an *closed set.*

(a) E is an open set: $\forall p \in E$, $\exists$ nbhd $\underline{N_r(p)} \subseteq E$

$N_r(p) = \{ y \in M : d(y, p) < r \}$

(b) F is a closed set: F contains all its limit points,

A limit point p of F is a point

s.t. $\forall r > 0$, $\exists q \in F$ s.t. $\begin{cases} q \neq p \\ q \in N_r(p) \end{cases}$

2. Answer the following questions.

   (a) (15 points) Prove that if E is open, then the set E^c, the complement of E, is closed.

(pf) E is open

$\Leftrightarrow$ $\forall p \in E$, $\exists$ nbhd $N_r(p) \subseteq E$.

$\Leftrightarrow$ $\forall p \in E$, $\exists$ nbhd $N_r(p)$ s.t. $N_r(p) \cap E^c = \phi$.

$\Leftrightarrow$ $\forall p \in E$. p is not a limit p of E^c

$\Leftrightarrow$ all limit points of E^c is in E^c

$\Leftrightarrow$ E^c is closed. #

   (b) (15 points) Suppose X is an open set and Y is a closed subset of X. Prove that $X \setminus Y$ is open.

(pf) Since Y is closed, Y^c is open by part (a).

$\Rightarrow$ $\underline{X \setminus Y = X \cap Y^c}$ = union of 2 open sets $\Rightarrow$ open

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$a \in X \setminus Y \Leftrightarrow \begin{cases} a \in X \\ a \notin Y \end{cases} \Leftrightarrow \begin{cases} a \in X \\ a \in Y^c \end{cases} \Leftrightarrow a \in X \cap Y^c$

3. (20 points) Suppose $S \subset \mathbb{R}$ such that S has an upper bounded. Prove that the supremum of S is a limit point or an element of S.

(pf) If $\sup S \in S$, then we are done.

If $\sup S \notin S$, for any $\underbrace{N_r(\sup S)}_{= (\sup S - r,\ \sup S + r)}$ with $r > 0$

$\Rightarrow \exists\, x \in N_r(\sup S)$ and $\sup S - r < x < \sup S$ since $\mathbb{Q}$ is dense in $\mathbb{R}$.

Since $x < \sup S$, $\Rightarrow \exists\, y \in S$ s.t. $y > x$.

$\Rightarrow y \in S$ and $y \in (x, \sup S) \subseteq N_r(\sup S)$

$y \neq \sup S$.

Hence, $\sup S$ is a limit point of S. #

4. Answer the following questions.

(a) (20 points) Prove that the intersection of finite open sets is open.

(pf) $\xi_1, \cdots, \xi_n$ are open.

$\forall p \in \xi_1 \cap \cdots \cap \xi_n = \bigcap_i \xi_i$, $\Rightarrow p \in \xi_i$ for $i = 1, \cdots, n$.

Since ξ_i is open, $\exists r_i > 0$ s.t. $N_{r_i}(p) \subseteq \xi_i$

Let $r = \min_i r_i$, then $r \leq r_i$ $\forall i$.

Then $p \in N_r(p) \underset{(r \leq r_i)}{\subseteq} N_{r_i}(p) \underset{\xi_i \text{ is open}}{\subseteq} \xi_i$ $\forall i$

Hence, $N_r(p) \subseteq \bigcap_i \xi_i$, $\Rightarrow \bigcap_i \xi_i$ is open. #

(b) (10 points) Consider the intersection of countably many open intervals defined as follows:

$$E = \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n} \right).$$

Is E an open set? Justify your answer.

$\forall x > 0$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < x$ $\Rightarrow x \notin (-\frac{1}{n}, \frac{1}{n})$

$\forall x < 0$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < -x$ $\Rightarrow x < -\frac{1}{n}$ $\Rightarrow x \notin (-\frac{1}{n}, \frac{1}{n})$

Hence, $E = \{0\}$.

But then $\forall$ nbhd $N_r(0)$, $\frac{r}{2} \in N_r(0)$, but $\frac{r}{2} \notin E$.

$\Rightarrow N_r(0) \nsubseteq E$, so E is not open.