Introduction to Quantitative Methods, Quiz 4

1. (30 points) Consider the following set

$$S = \{x \in \mathbb{C}\colon a_n x^n + \cdots + a_1 x + a_0 = 0 \text{ for some integers } a_n, \cdots, a_0 \text{ and } a_n \neq 0\}$$

   The set S is called the *algebraic numbers*. Prove that S is a countable set. (Hint: You can use the fact that any polynomial f with degree n has at most n roots.)

2. (20 points) Give the definition of a *metric space*.

3. Let Δ_n be the set $\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n \colon x_1 + \cdots + x_n = 1,\ x_i \geq 0 \text{ for all } 1 \leq i \leq n\}$. Let $p = (p_1, \cdots, p_n)$, $q = (q_1, \cdots, q_n) \in \Delta_d$, the *Hellinger distance* between p and q is defined as

$$H(p, q) = \frac{1}{\sqrt{2}} \sqrt{\sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2}$$

   Answer the following problems.

   a. (15 points) Let $x = (x_1, \cdots, x_n)$be any point in Δ_d, define the function $f : \Delta_n \to \mathbb{R}^n$ as:

$$f(x) = (\sqrt{x_1}, \cdots, \sqrt{x_n})$$

      What is $f(\Delta_n)$, the image of f?

   b. (15 points) Prove that (Δ_n) together with the distance function H is a metric space. You can use the fact that the Euclidean metric is a metric on $\mathbb{R}^n$ without a proof.

4. Given two metric spaces (X, d_X) and (Y, d_Y), a distance function $d_{X\times Y}$ on the Cartesian product $X \times Y$ can be defined as

$$d_{X\times Y}\left((x_1, y_1), (x_2, y_2)\right) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

   for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

   a. (15 points)

      Prove that $d_{X\times Y}$ is a metric on $X \times Y$

      Remark. This metric is called the *product metric*.

   b. (15 points) Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n, y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, the ℓ_1 distance between x and y is defined as

$$\ell_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

Use a. to prove that ℓ_1 distance is a metric on $\mathbb{R}^n$. (A proof without a. will get at most 10 points).

1. (30 points) Consider the following set

$$S = \{x \in \mathbb{C} \colon a_n x^n + \cdots + a_1 x + a_0 = 0 \text{ for some integers } a_n, \cdots, a_0 \text{ and } a_n \neq 0\}$$

The set S is called the *algebraic numbers*. Prove that S is a countable set. (Hint: You can use the fact that any polynomial f with degree n has at most n roots.)

Step 1: Prove that the set of "all polynomials of degree n with integer coefficients" is countable.

(pf) $\mathbb{Z}$ is countable. $\mathbb{Z}^2$ is countable.

$\Rightarrow$ By induction that $\mathbb{Z}^{n+1} = \mathbb{Z}^n \times \mathbb{Z}$ is countable.

Construct an injection

$$\mathbb{Z}^n[x] = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,\ a_i \in \mathbb{Z},\ a_n \neq 0\} \xhookrightarrow{f} \mathbb{Z}^{n+1}$$

by $a_n x^n + \cdots + a_0 \longmapsto (a_n, a_{n-1}, \cdots, a_0)$

Then, $|\mathbb{Z}^n[x]| = |f(\mathbb{Z}^n[x])| \leq \mathbb{Z}^{n+1}$.

Step 2: Prove the set of all polynomials with integer coefficients is countable.

(pf) $\mathbb{Z}[x] = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n[x]$, $\mathbb{Z}^n[x]$ is countable for all n.

$\Rightarrow \mathbb{Z}[x]$ is countable.

Step 3: S is countable.

(pf) For any polynomial $f(x)$, $S_f = \{x \in \mathbb{C} : f(x) = 0\}$ is the set of roots of f. $|S_f| \leq \deg f < \infty$ (finite)

$\Rightarrow S = \bigcup_{f \in \mathbb{Z}[x]} S_f$ is countable. #

2. (20 points) Give the definition of a *metric space*.

(See review notes.)

3. Let Δ_n be the set $\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n \colon x_1 + \cdots + x_n = 1, x_i \geq 0 \text{ for all } 1 \leq i \leq n\}$. Let $p = (p_1, \cdots, p_n)$, $q = (q_1, \cdots, q_n) \in \Delta_d$, the *Hellinger distance* between p and q is defined as

$$H(p,q) = \frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{n}(\sqrt{p_i} - \sqrt{q_i})^2}$$

Answer the following problems.

a. (15 points) Let $x = (x_1, \cdots, x_n)$be any point in Δ_d, define the function $f : \Delta_n \to \mathbb{R}^n$ as:

$$f(x) = (\sqrt{x_1}, \cdots, \sqrt{x_n})$$

What is $f(\Delta_n)$, the image of f?

$f : \Delta_n \longrightarrow \mathbb{R}^n$

$(x_1, \cdots, x_n) \longmapsto \underbrace{(\sqrt{x_1}, \cdots, \sqrt{x_n})}_{= (y_1, \cdots, y_n)}$

Then, $y_1^2 + \cdots + y_n^2 = \sum_{i=1}^{n} (\sqrt{x_i})^2 = \sum_{i=1}^{n} x_i = 1$

So, $f(\Delta_n) = \{ y = (y_1, \cdots, y_n) \in \mathbb{R}^n : \sum_{i=1}^{n} y_i = 1 \}$

b. (15 points) Prove that (Δ_n) together with the distance function H is a metric space. You can use the fact that the Euclidean metric is a metric on $\mathbb{R}^n$ without a proof.

Reparametriz $(x_1, \cdots, x_n)$ by $(\underbrace{f(x_1)}_{=f^{-1}(y_1)}, \cdots, \underbrace{f(x_n)}_{=f^{-1}(y_n)}) = (\underbrace{\sqrt{x_1}}_{=y_1^2}, \cdots, \underbrace{\sqrt{x_n}}_{=y_n^2})$,

we can define a distance function $\bar{H}$ on $f(\Delta_n)$ such that

$\bar{H}(\underbrace{f(p)}_{= x}, \underbrace{f(q)}_{= y}) = H(p, q)$ and (Δ_d, H) is a metric space.

$\Updownarrow$

$(f(\Delta_d), \bar{H})$ is a metric space.

$\Rightarrow \bar{H}(x, y) = H(f^{-1}(x), f^{-1}(y)) = H((x_1^2, \cdots, x_n^2), (y_1^2, \cdots, y_n^2))$

$= \frac{1}{\sqrt{2}} \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ for all $x, y \in \Delta_n$

So $\bar{H}$ is the Euclidean distance on $f(\Delta_n)$

$\Rightarrow (\bar{H}, f(\Delta_n))$ is a metric space. $\Rightarrow (H, \Delta_n)$ is a metric space.

4. Given two metric spaces (X, d_X) and (Y, d_Y), a distance function $d_{X\times Y}$ on the Cartesian product $X \times Y$ can be defined as

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

a. (15 points)

Prove that $d_{X\times Y}$ is a metric on $X \times Y$

Remark. This metric is called the *product metric.*

Need ① $d_{X*Y}\big((x,y),(x,y)\big) = 0$

(pf) Since (X, d_X), (Y, d_Y) are metric spaces, $\begin{cases} d_X(x,x) = 0 \\ d_Y(y,y) = 0 \end{cases}$

$\Rightarrow d_{X*Y}\big((x,y),(x,y)\big) = d_X(x,x) + d_Y(y,y) = 0+0 = 0$ #

② $d_{X*Y}\big((x_1,y_1),(x_2,y_2)\big) > 0$ if $(x_1,y_1) \neq (x_2,y_2)$

(pf) Say $x_1 \neq x_2$, then $d_X(x_1,x_2) > 0$.

$\Rightarrow \text{LHS} = \underbrace{d_X(x_1,x_2)}_{>0} + \underbrace{d_Y(y_1,y_2)}_{\geq 0} > 0$ #

③ $d_{X*Y}\big((x_1,y_1),(x_2,y_2)\big) = d_{X*Y}\big((x_2,y_2),(x_1,y_1)\big)$

(pf)

④ $d_{X*Y}\big((x_1,y_1),(x_2,y_2)\big) + d_{X*Y}\big((x_2,y_2),(x_3,y_3)\big)$

$\geq d_{X*Y}\big((x_1,y_1),(x_3,y_3)\big)$

(pf) Since $\begin{cases} d_X(x_1,x_2) + d_X(x_2,x_3) \geq d_X(x_1,x_3) \\ d_Y(y_1,y_2) + d_Y(y_2,y_3) \geq d_Y(y_1,y_3) \end{cases}$
+)

$\text{LHS} = \big[d_X(x_1,x_2) + d_Y(y_1,y_2)\big] + \big[d_X(x_2,x_3) + d_Y(y_2,y_3)\big]$

$\geq \big[d_X(x_1,x_3) + d_Y(y_1,y_3)\big] = \text{RHS}$ #

b. (15 points) Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n, y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, the ℓ_1 distance between x and y is defined as

$$\ell_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

Use a. to prove that ℓ_1 distance is a metric on $\mathbb{R}^n$. (A proof without a. will get at most 10 points).

We prove $l_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ is a metric space by induction on n.

① Base case: $l_1(x,y) = |x_1 - y_1|$ is a metric space on $\mathbb{R}$.

② Inductive step: If $\begin{cases} d_{\mathbb{R}^{n-1}}((x_1, \cdots, x_{n-1}), (y_1, \cdots, y_{n-1})) \text{ is a metric on } \mathbb{R}^{n-1} \\ d_{\mathbb{R}}(x_n, y_n) \text{ is a metric on } \mathbb{R}. \end{cases}$

$\to l_1$ distance on $\mathbb{R}^{n-1}$

$\to l_1$ distance on $\mathbb{R}$

Then, $d_{\mathbb{R}^n}((x_1, \cdots, x_n), (y_1, \cdots, y_n))$

$$= d_{\underbrace{\mathbb{R}^{n-1}}_{=X} \times \underbrace{\mathbb{R}}_{=Y}}\Big(\big(\underbrace{(x_1, \cdots, x_{n-1})}_{=a \in X}, \underbrace{x_n}_{=b \in Y}\big), \big(\underbrace{(y_1, \cdots, y_{n-1})}_{=c \in X}, \underbrace{y_n}_{=d \in Y}\big)\Big)$$

By (a), we have $d_{\mathbb{R}^n}$ to be a metric space on $\mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$

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