Introduction to Quantitative Methods, Quiz 4

1. (30 points) Consider the following set

$$S = \{x \in \mathbb{C} \colon a_n x^n + \cdots + a_1 x + a_0 = 0 \text{ for some integers } a_n, \cdots, a_0 \text{ and } a_n \neq 0\}$$

The set S is called the *algebraic numbers*. Prove that S is a countable set. (Hint: You can use the fact that any polynomial f with degree n has at most n roots.)

2. (20 points) Give the definition of a *metric space*.

3. Let Δ_n be the set $\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n \colon x_1 + \cdots + x_n = 1,\, x_i \geq 0 \text{ for all } 1 \leq i \leq n\}$. Let $p = (p_1, \cdots, p_n)$, $q = (q_1, \cdots, q_n) \in \Delta_d$, the *Hellinger distance* between p and q is defined as

$$H(p, q) = \frac{1}{\sqrt{2}} \sqrt{\sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2}$$

Answer the following problems.

a. (15 points) Let $x = (x_1, \cdots, x_n)$be any point in Δ_d, define the function $f : \Delta_n \to \mathbb{R}^n$ as:

$$f(x) = (\sqrt{x_1}, \cdots, \sqrt{x_n})$$

What is $f(\Delta_n)$, the image of f?

b. (15 points) Prove that (Δ_n) together with the distance function H is a metric space. You can use the fact that the Euclidean metric is a metric on $\mathbb{R}^n$ without a proof.

4. Given two metric spaces (X, d_X) and (Y, d_Y), a distance function $d_{X\times Y}$ on the Cartesian product $X \times Y$ can be defined as

$$d_{X\times Y}\left((x_1, y_1), (x_2, y_2)\right) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

a. (15 points)

Prove that $d_{X\times Y}$ is a metric on $X \times Y$

Remark. This metric is called the *product metric*.

b. (15 points) Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n, y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, the ℓ_1 distance between x and y is defined as

$$\ell_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

Use a. to prove that ℓ_1 distance is a metric on $\mathbb{R}^n$. (A proof without a. will get at most 10 points).