Introduction to Quantitative Methods, Quiz 3

1. (20 points) If z is a complex number, prove that there exist an real number $r \geq 0$ and a complex number $|w| = 1$, such that $z = rw$. Are r, w always uniquely determined by z ?.

2. (20 points) The distributive law says that for all real numbers c, a_1 and a_2, we have $c(a_1 + a_2) = ca_1 + ca_2$. Use this law and mathematical induction to prove that, for all natural numbers $n > 2$, if $c, a_1, a_2, ..., a_n$ are real numbers, then

$$c(a_1 + \cdots + a_n) = ca_1 + \cdots + ca_n$$

3. (15 points) A field $(F, +, \times)$ is an *ordered field* together with a order $<$ if the order satisfies the following properties for all $a, b, c \in F$.

   - if $a \leq b$ then $a + c \leq b + c$
   - if $0 \leq a$ and $0 \leq b$, then $0 \leq ab$

   Prove that there is no order $<$ such that $\mathbb{C}$ together with $<$ is not a ordered field. (Hint: $x^2 \leq 0$ for all x in an ordered field)

4. (20 points) Let $z_1 \cdots z_n$ be complex numbers. Prove that $|z_1 \cdots z_n| = |z_1 \cdots z_n|$. (Hint: You can prove the case that $n = 2$ first, then extend it to any natural number n.)

5. Let $x, y \in \mathbb{C}^k = \{(z_1, \cdots, z_n)\}$ be two vectors in the complex space.

   a. (20 points) The *Cauchy-Schwarz inequality* state that the following inequality holds:

   $$|\langle x, y\rangle| \leq |x|\,|y|$$

   , where $\langle x, y\rangle = \sum_{i=1}^{k} x_i \bar{y}_i$ and $\bar{y}_i$ is the complex conjugate of y_i. Prove this inequality.

   b. (20 points) State and prove the the *triangle inequality*:

1. (20 points) If z is a complex number, prove that there exist an real number $r \geq 0$ and a complex number $|w| = 1$, such that $z = rw$. Are r, w always uniquely determined by z ?.

Proof: (1) If $z = 0$, then $r = 0$, $w = \pm 1$ satisfy $z = rw$ and $r \geq 0$, $|w| = 1$.

The number w is not unique.

(2) If $z \neq 0$, let $r = |z|$ and $w = \frac{z}{|z|}$,

, then $z = rw$, $r > 0$ and $|w| = \left|\frac{z}{|z|}\right| = \frac{|z|}{|z|} = 1$, (r, w) is a solution.

If $z = rw$, then $\bar{z} = \overline{rw} = r\bar{w}$,

$|z|^2 = z \cdot \bar{z} = r^2 w \cdot \bar{w} = r^2 |w|^2 = r^2$

Therefore, $r = |z|$ and $w = \frac{z}{|z|}$ is the unique solution.

2. (20 points) The distributive law says that for all real numbers c, a_1 and a_2, we have $c(a_1 + a_2) = ca_1 + ca_2$. Use this law and mathematical induction to prove that, for all natural numbers $n > 2$, if $c, a_1, a_2, ..., a_n$ are real numbers, then

$$c(a_1 + \cdots + a_n) = ca_1 + \cdots + ca_n$$

Proof: We prove it by inductions on $n \geq 2$

- For the base case $n=2$, it is implied by the distributive law.
- For the inductive step, we assume that for any real numbers $a_1, \cdots, a_n$ and real number c, $c(a_1 + \cdots + a_n) = ca_1 + \cdots + ca_n$.

Consider any real numbers $a_1, \cdots, a_{n+1}$, we have

$$\begin{aligned} c(a_1 + \cdots + a_{n+1}) &= c((a_1 + \cdots + a_n) + a_{n+1}) \\ &= c(a_1 + \cdots + a_n) + ca_{n+1} && \text{(distributive law)} \\ &= ca_1 + \cdots + ca_n + ca_{n+1} && \text{(induction hypothesis)} \end{aligned}$$

Therefore $c(a_1 + \cdots + a_n) = ca_1 + \cdots + ca_n$ for all $n \geq 2$.

3. (15 points) A field $(F, +, \times)$ is an *ordered field* together with a order $<$ if the order satisfies the following properties for all $a, b, c \in F$.

- if $a \leq b$ then $a + c \leq b + c$
- if $0 \leq a$ and $0 \leq b$, then $0 \leq ab$

Prove that there is no order $<$ such that $\mathbb{C}$ together with $<$ is not a ordered field. (Hint: $x^2 \geq 0$ for all x in an ordered field)

Proof: If $(F, +, \times)$ together with a field is an ordered field, we claim that $x^2 \geq 0$ holds for all $x \in F$

(pf) For any $a, b \in \mathbb{F}$, if $0 \leq a$ $0 \leq b$, then $0 \leq ab$ $\cdots\cdots$ $(*)$

Since $<$ is an order, one of the inequalities $0 < x$, $0 = x$, $0 > x$ holds.

(1) If $0 = x$, then $x^2 = 0$

(2) If $0 \leq x$, then $0 \leq x^2$ by choosing $a = b = x$ in $(*)$

(3) If $0 \geq x$, then $0 \leq x^2$ by choosing $a = b = -x$ in $(*)$ #

If there exist an order $>$ such that $\mathbb{C}$ together with $>$ is an ordered field. We have $-1 = i^2 > 0$, contradict with $1 = 1^2 > 0$.
So there does not exist such order.

4. (20 points) Let $z_1 \cdots z_n$ be complex numbers. Prove that $|z_1 \cdots z_n| = |z_1| \cdots |z_n|$. (Hint: You can prove the case that $n = 2$ first, then extend it to any natural number n.)

Proof: We prove $|z_1 \cdots z_n| = |z_1| \cdots |z_n|$ for all $n \geq 2$ by strong induction on n.

- For base case $n=2$, let $z_1 = a+bi$, $z_2 = c+di$ with $a,b,c,d \in \mathbb{R}$, then $z_1 z_2 = (ac-bd)+(ad+bc)i$
  We have

$$\begin{aligned} |z_1 z_2|^2 &= (ac-bd)^2+(ad+bc)^2 \\ &= a^2c^2+b^2d^2+2abcd+a^2d^2+b^2c^2-2abcd \\ &= (a^2+b^2)(c^2+d^2) \\ &= |z_1|^2|z_2|^2 \end{aligned}$$

  Since $|z_1z_2|, |z_1|, |z_2| \geq 0$, we have $|z_1z_2| = |z_1||z_2|$.

- For inductive step, we assume that $|z_1 \cdots z_k| = |z_1| \cdots |z_k|$ holds for all $k \leq n$

  For any complex numbers $z_1, \cdots, z_{n+1}$, we have

$$\begin{aligned} |z_1 \cdots z_{n+1}| &= |(z_1 \cdots z_n) z_{n+1}| \\ &= |z_1 \cdots z_n| \, |z_{n+1}| \quad (\text{induction hypothesis for } k=2) \\ &= |z_1| \cdots |z_n| \, |z_{n+1}| \quad (\text{induction hypothesis for } k=n) \end{aligned}$$

Therefore $|z_1 \cdots z_n|$ holds for any complex numbers $z_1, \cdots z_n$ and $n \geq 2$.

5. Let $x, y \in \mathbb{C}^k = \{(z_1, \cdots, z_n)\}$ be two vectors in the complex space.

a. (20 points) The *Cauchy-Schwarz inequality* state that the following inequality holds:

$$|\langle x, y\rangle| \le |x|\,|y|$$

where $\langle x, y\rangle = \sum_{i=1}^{k} x_i \bar{y}_i$ and $\bar{y}_i$ is the complex conjugate of y_i. Prove this inequality.

b. (20 points) State and prove the the *triangle inequality*:

Proof: (a.)

For all $t \in \mathbb{C}$, we have

$$0 \le \|x - ty\|^2$$
$$= \langle x - ty, x - ty\rangle$$
$$= \langle x, x\rangle - t\langle y, x\rangle - \bar{t}\langle x, y\rangle + |t|^2\langle y, y\rangle$$

Choose $t = \frac{\langle x, y\rangle}{\langle y, y\rangle}$, then the inequality become

$$0 \le \langle x, x\rangle - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle}$$

$$\Rightarrow \quad 0 \le |x|^2|y|^2 - |\langle x, y\rangle|^2$$

$$\Rightarrow \quad |\langle x, y\rangle| \le |x|\,|y|$$

(b.) The triangle inequality is $|x+y| \le |x| + |y|$.

Proof: Let $\langle x, y\rangle = x \cdot \bar{y} = \sum_{i=1}^{n} x_i \bar{y}_i$

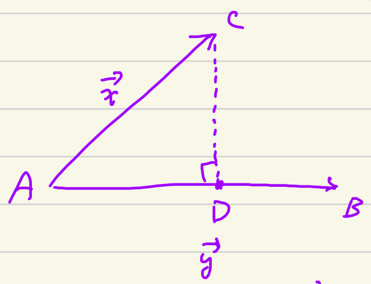
We have

$$|x+y|^2 = (x+y)\cdot(\bar{x}+\bar{y})$$
$$= x\cdot\bar{x} + x\cdot\bar{y} + \bar{x}\cdot y + y\cdot\bar{y}$$
$$= |x|^2 + |y|^2 + 2\,Re(\langle x, y\rangle)$$
$$\le |x|^2 + y^2 + 2|\langle x, y\rangle| \quad (Re(z) \le |z| \text{ for all } z)$$
$$\le |x|^2 + |y|^2 + 2|x|\cdot|y| \quad (\text{Cauchy inequality in } 5.(a))$$
$$= (|x| + |y|)^2$$

Since $|x+y|, |x|, |y| \ge 0$, we have $|x+y| \le |x| + |y|$ for all complex numbers x, y.

Alternative proof of (a):

Alternatively, consider $\vec{AB} = \vec{y}$, $\vec{AC} = \vec{x}$



By Pythagorean Theorem,

$|\vec{AC}|^2 = |\vec{AD}|^2 + |\vec{CD}|^2$,

So $|\vec{AC}|^2 \ge |\vec{AB}|^2$

Claim: $\vec{AD} = \frac{\langle \vec{x}, \vec{y}\rangle}{\langle \vec{y}, \vec{y}\rangle} \cdot \vec{y}$

Then $\underbrace{|\vec{AC}|^2}_{\langle \vec{x}, \vec{x}\rangle} \ge |\vec{AD}|^2 = \frac{|\langle x, y\rangle|^2}{\langle \vec{y}, \vec{y}\rangle}$

(pf) Let $\vec{AD} = k\cdot\vec{y}$,

$\vec{AC}\cdot\vec{AB} = (\vec{AD} + \vec{DC})\cdot\vec{AB}$

$\underbrace{\vec{AC}}_{\vec{x}} \cdot \underbrace{\vec{AB}}_{\vec{y}} = \underbrace{\vec{AD}}_{k\cdot\vec{y}} \cdot \underbrace{\vec{AB}}_{\vec{y}} + 0 \quad (\because \vec{DC} \perp \vec{AB})$

$\Rightarrow \vec{x}\cdot\vec{y} = (k\cdot\vec{y})\cdot\vec{y} \Rightarrow k = \frac{\langle \vec{x}, \vec{y}\rangle}{\langle \vec{y}, \vec{y}\rangle}$ #