Introduction to Quantitative Methods, Quiz 3

1. (20 points) If z is a complex number, prove that there exist an real number $r \geq 0$ and a complex number $|w| = 1$, such that $z = rw$. Are r, w always uniquely determined by z ?.

2. (20 points) The distributive law says that for all real numbers c, a_1 and a_2, we have $c(a_1 + a_2) = ca_1 + ca_2$. Use this law and mathematical induction to prove that, for all natural numbers $n > 2$, if $c, a_1, a_2, ..., a_n$ are real numbers, then

$$c(a_1 + \cdots + a_n) = ca_1 + \cdots + ca_n$$

3. (15 points) A field $(F, +, \times)$ is an *ordered field* together with a order $<$ if the order satisfies the following properties for all $a, b, c \in F$.

   - if $a \leq b$ then $a + c \leq b + c$
   - if $0 \leq a$ and $0 \leq b$, then $0 \leq ab$

   Prove that there is no order $<$ such that $\mathbb{C}$ together with $<$ is not an ordered field. (Hint: $x^2 \geq 0$ for all x in an ordered field)

4. (20 points) Let $z_1 \cdots z_n$ be complex numbers. Prove that $|z_1 \cdots z_n| = |z_1||z_2| \cdots |z_n|$. (Hint: You can prove the case that $n = 2$ first, then extend it to any natural number n.)

5. Let $x, y \in \mathbb{C}^k = \{(z_1, \cdots, z_n)\}$ be two vectors in the complex space.

   a. (20 points) The *Cauchy-Schwarz inequality* state that the following inequality holds:

   $$|\langle x, y \rangle| \leq |x|\,|y|$$

   where $\langle x, y \rangle = \sum_{i=1}^{k} x_i \bar{y}_i$ and $\bar{y}_i$ is the complex conjugate of y_i. Prove this inequality.

   b. (20 points) State and prove the the *triangle inequality*: