Introduction to Quantitative Methods, Quiz 2

1. Let $\mathbb{N}$ denote the set of positive integers. Answer the following questions.
   a. (20 points) Give the definitions of least upper bound of a set $S \subset \mathbb{R}$.
   b. (20 points) What is the value of $\inf\{\frac{1}{x^n} : n \in \mathbb{N}, x \in \mathbb{R} \text{ and } x > 1\}$
2. (30 points) Let $A \subseteq B \subseteq \mathbb{R}$, prove that $\sup A \leq \sup B$ and $\inf A \geq \inf B$
3. Recall that the mathematical construction of a real number is a cut. A cut α is a subset of $\mathbb{Q}$ satisfying the following conditions.
   (1) $\alpha \neq \emptyset, \mathbb{Q}$
   (2) If $p \in \alpha, q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$ (closed downwards).
   (3) If $p \in \alpha$, then there is a $q \in \alpha$ such that $p < q$ (no largest number).

   Answer the following questinos:
   a. (30 points) If a relation $\leq$ defined on cuts is defined such that $\alpha < \beta$ if and only if $\alpha \subsetneq \beta$ for any two cuts α, β. Show that $<$ is an order on cuts.
   b. (20 points) If p is a rational number and $p^2 < 2$, let $q = \frac{2p+2}{p+2}$. Show that $p < q$ and also $q^2 < 2$. Then deduce that the set $\{x \in \mathbb{Q} : x < 0\} \cup \{x \in \mathbb{Q} : x^2 < 2\}$ is a cut.