Introduction to Quantitative Methods, Quiz 10

1. (20 points) State the *Intermediate Value Theorem*.

2. (20 points) Let $f(x) = \begin{cases} -x, & x \in \mathbb{Q} \\ x, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Find the set $\{x \in \mathbb{R} \colon f \text{ is discontinuous at } x\}$.

3. For each of the following functions, determine whether or not the function f is continuous at a given value a. If it is not continuous, decide whether $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist.

   (a) (15 points) $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$, $a = 0$.

   (b) (15 points) $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 1, & x = 0 \end{cases}$, $a = 0$.

4. Let (X, d_1) and (Y, d_2) be two metric space. A function $f \colon X \to Y$ is called *Lipschitz continuous* (on X) if there is a constant c such that $d_2\big(f(x), f(y)\big) \leq c \cdot d_1(x, y)$ for all $x, y \in X$ and c is independent of x and y.

   (a) (10 points) Prove that $f(x) = x^2$ is Lipschitz continuous on $[0, 1]$.

   (b) (10 points) Prove that if a function g is Lipschitz-continuous on X, then it is uniformly continuous on X.

   (c) (10 points) By the results in 4.a and 4.b, we know that $f(x) = x^2$ is uniformly continuous on $[0, 1]$. Is f uniformly continuous on the whole real line $\mathbb{R}$? Prove your answer (you can do this problem without proving 4.a and 4.b).