Introduction to Quantitative Method: Midterm

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Problem 1.

(a) There are three things we need to check:

- **Reflexivity:** $A \sim A$. So $>$ is not an equivalence relation.
- **Symmetry:** $A \sim B \Rightarrow B \sim A$. So $\subset$ is not an equivalence relation.
- **Transitivity:** If $A \sim B$ and $B \sim C$ then $A \sim C$. So *(ii)* is not an equivalence relation.

(b)
  i. It's true, but you need to show it carefully. Many students misuse $\sim$ and $=$, and regarding that $f(a) = f(b) \Rightarrow a = b$, so it's trivial a equivalence relation.

  ii. It's not true, since empty set don't equivalent to itself and the transitivity doesn't hold by construction.[1]

Grading Scheme:

Most students received full credit for this problem. However, some students confused the concepts of **equivalence relation** and **trichotomy**... Bye-bye! If you completely wrote the wrong answer (e.g., writing "False" for *(i)*), then you received 0 points.[2]

[1] Take $A = \{1, 2\}, B = \{2, 3\}, C = \{3, 4\}$ is obvious example.

[2] **Murmur**: Why are there students who check everything correctly yet write the wrong conclusion?

Problem 2.

(a) In *Rudin*, **'countable'** means "there exists a bijection between $\mathbb{N}$ and the set." But in some books, it means "there exists an **injection** into $\mathbb{N}$." Both of them can receive full credit.

(b)

i. $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable, if $\mathbb{R} - \mathbb{Q}$ is countable, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$, the union of countable sets, is countable, which is a contradiction. If you missed the last argument, you will miss 1 point here.

ii. It's clear, since $(a, b) \sim (-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbb{R}$, where $\sim$ denotes equivalence in the sense or cardinality. However, you need to explicitly write down these bijections: $f : x \to \frac{-\pi}{2(a-b)}(x - b) + \frac{\pi}{2(b-a)}(x - a)$ and $\tan(x)$, or at least mention them. Or you could write "WLOG" (without loss of generality).[3]

Grading Scheme:

Wrong conclusion: 0 point. **Without detail or just claim an answer**: -1 or -2 points.

[3]The other method is to write down the *Cantor's diagonal argument*(with detail, don't just mention it).

Problem 3.

(a) There are also three things we need to check for a set X and a distance function d. Let $x, y, z \in X$:

- **Positivity:** $d(x, y) \geq 0$ and the equality holds if and only if $x = y$
- **Symmetry:** $d(x, y) = d(y, x)$.
- **Triangle Inequality:** $d(x, y) \leq d(x, z) + d(z, y)$

(b) i. $d_1 = 0$ when $x = 1$ and $y = -1$; hence, the first condition does not hold. Thus, d_1 is not a metric. No partial credit will be given for other answers.

ii. This is a metric. The first two conditions hold trivially (though you should verify this). The triangle inequality is the crucial part of this metric. If you claim it holds without showing any effort, you will lose 2 points.

Rewrite the problem: "*Prove* $\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}$ *given that* $a \leq b + c$, *where* $a = |x - y|$, $b = |y - z|$, *and* $c = |z - x|$." [4]

There are many ways to prove this; here we use a brute-force method (expand it):

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c} \iff a \leq b + bc + c + cb + abc$$

This inequality is true given the condition. The other one sophisticated way is noticing that $\frac{x}{1+x}$ is an increasing function, since $a \leq b + c$,

$$\frac{a}{1+a} \leq \frac{b+c}{1+b+c} = \frac{b}{1+b+c} + \frac{c}{1+b+c} \leq \frac{b}{1+b} + \frac{c}{1+c}$$

,as desired.

Grading Scheme

- If you write to more useless and wrong words, you may loss some credits.
- If you claim an answer without any statement, then you will get at most 2/4 points.

[4] The given condition follows from $|\cdot|$ being a metric on $\mathbb{R}$.

Problem 4.

(a) All points in S are interior points. An interior point means that if $x \in S$, then $\exists r > 0$ such that $B_r(x) \subset S$ (i.e., a neighborhood **of x** is completely contained in S).

(b) i. Yes, it is. For each point $P(x, y) \in A$, we can take $r = \frac{|x+2y-0|}{\sqrt{5}}$ and $B_r(P) \subset A$, so A is open. Those who only say "*let r be the distance between P and the line $x + 2y = 0$*" will also receive full credit (4/4 points).

ii. Yes, it is. For every $x \in B \subset (0, 1)$, there is a smallest $n \in \mathbb{N}$ such that $\frac{1}{n} < x$. To show that there is a neighborhood of x that is contained in B, we can take

$$r = 0.5 \cdot \min \left\{ \left| x - \frac{1}{n} \right|, \left| x - \frac{1}{n-1} \right| \right\}$$

,as desired.

In fact, $B = \bigcup_{n=1}^{\infty} (\frac{1}{n}, \frac{1}{n+1})$(verify it by yourself) is a union of open sets, which is still open.

Grading Scheme

- As P3 shows: wrong answer get 0 point and a claim without proof get 2 over 4 points.
- Some students use wrong implication or there are some ambiguity in their arguments, may loss some points. For example, *countable union of closed sets is closed*, is $\{a_n = 1 - \frac{1}{n} | n \in \mathbb{N}\}$ a joke?

Problem 5.

This problem has many solutions and perspectives:

1. The complement of an open set is closed. Since a singleton $\{x\}$ is closed, and a finite union of closed sets is still closed, every subset in M is closed. Moreover, every set is the complement of its complement, hence every subset in M is also open.

2. Let $r = 0.5 \cdot \min\{d(x, y) \mid x, y \in M\}$.[5] For all $x \in M$, there is an r-neighborhood, $B_r(x)$, whose interior $\{x\}$ is contained in $\{x\}$. Hence, a singleton is open. Since an arbitrary union of open sets is open, every subset in M is open.

   Because a finite set has no limit point (i.e., the set of limit points of a subset in M is empty)(we use the definition in *Rudin*), every subset satisfies the definition of closedness. As a result, every set in M is open and closed at the same time.

Grading Scheme

- Checking that "the complement of an open set is closed and vice versa" is good, but it will not earn extra points.
- You may combine some arguments together, and I will give full credit as long as the explanation is written properly.

Problem 6.

Suppose we can find such a set $\{x_1, x_2, \ldots, x_n\}$ and express y uniquely as $\sum_{i=1}^{n} r_i x_i$ where $r_i \in \mathbb{Q}$. Then there would be an intrinsic bijection between $\mathbb{R}$ and $\mathbb{Q}^n$, mapping y to $(r_1, r_2, \ldots, r_n)$.

However, we know that $\mathbb{R}$ is uncountable, whereas $\mathbb{Q}^n$, being a finite product of the countable set $\mathbb{Q}$, is countable. Since there cannot be a bijection between sets of different cardinalities, this leads to a contradiction. Therefore, it is impossible to find a set $\{x_1, x_2, \cdots, x_n\}$ such that y can be uniquely expressed with coefficients in $\mathbb{Q}$.

Grading Scheme

- Say that the set is a finite dimensional vector space over $\mathbb{Q}$ is ok(?).
- Only say that $\mathbb{R}$ is uncountable is worth at most 2 points.
- Any mistake -1-1-1 $\sim$

[5] Elements are finite, so the set of distances is finite, and we can take the minimum.

Problem 7.

(i) Consider the set $A = \{(a, b) : q \in (a, b) \subset S\}$. This set is nonempty since S is open, meaning there exists a neighborhood of q in S, specifically $B_r(q) \subset S$. Define $l_q = \inf A$ and $r_q = \sup A$. By the construction of l_q and r_q, every interval (a, b) containing q satisfies $l_q < a$ and $b < r_q$. We refer to (l_q, r_q) as the **maximal interval in S that contains q**.[6]

(ii) Let (l_p, r_p) and (l_q, r_q) be two intervals that have some common element. Suppose that $l_q \neq l_p$ or $r_q \neq r_p$. Without loss of generality, assume that $l_q < l_p < r_q$. Since all points in (l_q, r_q) belong to S, we can extend the interval (l_p, r_p) to (l_q, r_p), which is a open interval contains p and is still within S. This extension contradicts the maximality of (l_p, r_p). Therefore, we must have $l_q = l_p$, and similarly, $r_q = r_p$.

(iii) Let J be an arbitrary open set in $\mathbb{R}$. From (ii), we can define an equivalence relation on elements in J: $a \sim b$ **if** $l_a = l_b$. The density of $\mathbb{Q}$ implies that every element in J must be equivalent to some rational number.

Using (ii) again, we see that J can be decomposed into a disjoint union of intervals (l_{q_i}, r_{q_i}), where $q_i \in \mathbb{Q}$ is the representative of each equivalence class. Finally, the countability of this union is guaranteed by the countability of $\mathbb{Q}$. *Q.E.D.*

Grading Scheme

- Many students struggle with the notation here, especially when it involves $l_q = -\infty$ or $r_q = \infty$, and then proceed to utter some truly ”amazing” mathematical statements... um-mmmm...
- The **maximality of the definition of l_q and r_q** is a crucial observation in this problem. Concepts like sup and inf are essential tools to concretely establish this idea.

[6] Otherwise, we could extend (l_q, r_q) to obtain a *larger* interval, contradicting the definitions of l_q and r_q.

Problem 8.

(a) First, note that $d_1 \leq d_2$ by directly squaring both sides:

$$\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - y_i)^2 + 2 \cdot \sum_{i \neq j} |x_i - y_i| \cdot |x_j - y_j|$$

and taking $c_1 = 1$, as desired.

On the other hand, we can use either *Jensen's inequality* or *Cauchy's inequality* to show that $c_2 = \sqrt{n}$ is sufficient. (I state this here and will prove it in the following section.)

Jensen: the convexity of square root function leads that

$$\frac{f(\sum x_i)}{n} \leq f(\frac{\sum x_i}{n}) \Rightarrow \frac{\sum \sqrt{(x_i - y_i)^2}}{n} \leq \sqrt{\frac{\sum(x_i - y_i)^2}{n}} \Rightarrow \sum |x_i - y_i| \leq \sqrt{n} \cdot \sqrt{\sum(x_i - y_i)^2}$$

Cauchy: clearly,

$$\sum |x_i - y_i| = \sum |x_i - y_i| \cdot 1 \leq \sqrt{\sum(x_i - y_i)^2} \cdot \sqrt{\sum 1^2} = \sqrt{\sum(x_i - y_i)^2} \cdot \sqrt{n}$$

(b) It's a pretty common problem, and you can google "share the same topology" by yourself.[7]

Grading Scheme

- I think many of you might struggle with the concept of 'global'. Here, 'global' means that c_1 and c_2 are not functions of x or y, and can be used uniformly between two metric spaces.
- Those who say Archimedean property will get 0 points in (a).
- (b) need some detailed estimation, and I'm too lazy to do it $\sim$ XD.

[7] Refer to: , which is a example with the metric in 3.(b).(ii) or this link.