Introduction to Quantitative Methods, Final Exam 2

1. (10 points) Give the statement of *the Heine-Borel theorem.*

2. Define the discrete metric as

$$d(x,y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

   (a) (5 points) Verify that the discrete metric is indeed a metric.

   (b) (5 points) Is $\mathbb{N}$ compact with the discrete metric ?

3. (a) (10 points) Let $d(x,y)$ be the Euclidean distance on $\mathbb{R}^n$. Prove that for any $y \in \mathbb{R}^n$, the function $f_y(x) = d(x,y)$ is continuous. You get 5 points if you prove the case when $n = 1$ and $d(x,y) = |x - y|$.

   (b) (10 points) Prove that the function $f(x) = x^2$ is **not** uniformly continuous on $\mathbb{R}$.

4. (a) (10 points) Let $\{x_n\}$ be a sequence in a metric space X. Prove that if $\{x_n\}$ converges, then $\{x_n\}$ is a Cauchy sequence.

   (b) (10 points) Let $y_n = \frac{2n^2}{1+n^2}$ for all $n \in \mathbb{N}$. Prove that the limit of $\{y_n\}$ exists.

   (c) (10 points) Let $x_1 = 4$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{4}{x_n}\right)$ for all $n \in \mathbb{N}$. Prove that the limit of $\{x_n\}$ exists.
   Hint: You can try to prove $|x_{n+1} - 2| \leq \frac{1}{2}|x_n - 2|$ first.

5. (a) (10 points) Let K be a compact set and F be a closed subset of K. Prove that F is compact.

   (b) (10 points) Let X, Y be metric spaces, $f\colon X \to Y$ is a function. Prove that X is continuous if and only if for any open subset U of Y, the set $f^{-1}(U) = \{x \in X\colon f(x) \in U\}$ is open.

   (c) (10 points). Let X, Y be metric spaces, $f\colon X \to Y$ is a bijective function and $f^{-1}\colon Y \to X$ is its inverse. Prove that if f is continuous and X is compact, then f^{-1} is continuous.