Is Last Minute Bidding Bad?

Joseph Tao-yi Wang*

Humanities and Social Sciences, Caltech

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Abstract

We demonstrate how last minute bidding on internet auctions is not compatible with a private value setting with a proxy bidding system, but can be rationalized by adding another identical auction. In the repeated eBay auction model, the last minute bidding equilibrium, in which bidders only bid at the last minute in the first auction, is the unique symmetric equilibrium considering monotonic, undominated strategies. Nonetheless, though high valuation bidders efficiently win the items, under strict affiliation, expected revenue is lower in the first auction than in the last, and sellers can increase their revenue by running a joint multi-unit auction. (JEL C73, D44, L81)

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Pluralitas non est ponenda sine necessitate.

Plurality shouldn't be posited without necessity.

—William of Occam (1285–1349)

During the past ten years, online auction houses such as eBay help millions of buyers and sellers meet and trade with each other. One interesting phenomenon of eBay auctions that has not been fully understood is "sniping," or last minute bidding, meaning that most buyers submit their bids very late during these online auctions.[1] As Patrick Bajari and Ali Hortacsu (2003) note, "more than 50 percent of final bids are submitted after 90 percent of the auction duration has passed."

Intuitively, if there were overlapping auctions, it would be natural to see people bid on the auctions that end first, and then bid on the next auction if they did not win the first one. This would result in bids coming only after all previous auctions are gone, and hence, "late." However, sniping refers not to "late" bidding, but attempts to bid almost exactly on the last second. In fact, attempting to bid at the last second is a common practice on all kinds of items. There are even various specialized softwares online that place last minute bids for you.[2]

The standard auction theory with private value (PV) has little to say about last minute bidding. Since auction houses like eBay have proxy bidding systems, buyers submit a proxy bid as their maximum and the eBay robot matches new incoming bids up to this maximum. Hence, setting a proxy bid of one's valuation as soon as he or she sees the auction would be a dominant strategy.[3] Nevertheless, last minute bidding is commonly observed in all categories of items, common value and private value alike.

Moreover, most bidders rationalize last minute bidding saying it can "avoid triggering a price war between each other." On the contrary, standard auction theory tells

[1] See Alvin E. Roth and Axel Ockenfels (2002) for a full description of this phenomenon.

[2] For example, a keyword search of "snipe" on google will give you sponsor links to www.ezsniper.com, www.bidnapper.com, www.auctionsniper.com, www.powersnipe.com, etc.

[3] This is why Bajari and Hortacsu (2003) think it is crucial private value models (such as Ockenfels and Roth (2006)) require additional assumptions such as lost last minute bids.

us that the timing of the proxy bids does not matter as long as everyone bids their valuation, and hence, such common sense rationale should not hold.[4] Interesting enough, the common sense rationale assumes the possibility of a price war if bidders bid early, hinting that bidders would "revise" their valuation and bid aggressively if someone else bids early, and would not do so if they bid on the last minute. This raises the question: Is this simply a misperception, or is there something that is not captured in the standard model?

On the other hand, it would not be surprising to see similar or identical items sold *repeatedly* or *sequentially*[5] for auction houses like eBay where thousands of items are auctioned off daily. If there are two identical auctions conducted in a row, bidders need not want to bid up to their true valuation in the first auction if they expect to win the second one at a lower price. Hence, their maximum willingness to bid for the current auction would not equal to their true valuation. In fact, it would be the expected winning price of the next auction, which is the expectation of the highest loser's valuation in the next auction, conditioning on being the tied winner of the current one. Since this expectation is correlated with other bidders' valuations, it creates a "common value component" in the private value setting, and brings in all of the information issues we might face in a common value environment.[6]

The idea of identical auctions conducted simultaneously or repeatedly is not new. Paul R. Milgrom and Robert J. Weber (2000) consider three different types of simultaneous auctions and four sequential (or repeated) ones. Under the affiliated private value (APV) setting, they proved that the winning price path of sequential first price

[4]In fact, with a minimum increment, bidders would actually prefer to bid as early as possible.

[5]A "repeated auction" is a sequence of identical single-unit auctions conducted repeatedly, and is what Milgrom and Weber called "sequential auctions." Here, we use the terms "repeated auction" and "sequential auction" interchangeably. However, note that the term "sequential auction" might also refer to other types of auction designs, such jump bidding followed with a button auction in Christopher Avery (1998).

[6]In fact, this is why repeated auctions act similar to common value auctions, and as shown below, induce last minute bidding behavior.

auctions with price revelation follow a martingale.[7] Robert J. Weber (1983) characterized the equilibria of various sequential auctions under the independent private value (IPV) assumption and proved the revenue equivalence theorem behind these auctions. Orley Ashenfelter (1989) describes repeated wine auctions which produced decreasing prices within seconds apart, and R. Preston McAfee and Daniel Vincent (1993) show how increasing risk aversion can create this so called, "decreasing price anomaly." However, most literature on empirical auctions abstract away from repetition and stick to the single auction model. Although repetition may not be important in traditional auctions, it should play an important role in internet auctions since an easy search tool could locate hundreds or even thousands of similar auction conducted repeatedly.

In this paper we follow the repeated auctions literature and construct a repeated auction model to explain last minute bidding in eBay auctions. In repeated eBay auctions with affiliated private value (APV), bidders only bid at the last minute and shade their bids in the early auction. However, the increasing equilibrium results in the highest bidders winning the auctions, and hence, the ending rule of eBay does not hurt efficiency. On the other hand, with "strict" affiliation, the expected revenue of the first auction is less or equal to that of the last,[8] in which equality is maintained when valuations are, say, independent. Moreover, expected revenue of the last auction would equal to the unit price in a multi-unit Vickrey auction (selling the two items simultaneously). Therefore, the seller of the first auction would be better off to conduct her auction later or together with the other seller when there is "strict" affiliation. However, such revenue ranking is eliminated when valuations are independent, and revenue equivalence, as shown in Weber (1983) is restored.

In contrast, we show that without repetition, neither proxy bidding nor the fixed

[7]They also showed that the repeated button auction would lead to full information revelation, and exact same winning prices. Unfortunately, their conjectures about other sequential auctions could not be verified. See their own discussion in Milgrom and Weber (2000).

[8]This revenue difference is closely related to the declining price anomaly described in Ashenfelter (1989), and McAfee and Vincent (1993) provide some possible explanation.

ending time per se, could lead to last minute bidding behavior we actually observe. Hence, it is indeed the case that "plurality is introduced only when necessary."[9]

There are other various alternative explanations of last minute bidding. In the common value setting, Axel Ockenfels and Alvin E. Roth (2006) showed an example in antique auctions where amateurs would bid after experts who have more information, and thus inducing the experts to bid at the last minute to avoid facing a bidding war against the amateurs. Julia Schindler (2003) elaborates on this idea of withholding information and shows how different interdependent value structures can induce last minute bidding. These are all under the standard auction framework.

In the private value setting, however, additional assumptions are needed to induce last minute bidding. For example, Ockenfels and Roth (2006) add the assumption that the ending period has a positive probability p that a last minute bid is not successfully transmitted. With the probability p as common knowledge, they showed another example of an equilibrium where both bidders jump in at the last minute bidding their valuations and hoping that only their own bids are successfully transmitted and the other's bid is lost. Since there is a positive probability of losing some bids, sellers' expected revenue, and hence, eBay's service charges, are lowered. Moreover, overall efficiency is not always obtained since there is a positive probability that the highest bid is not transmitted. In this case, one might wonder why auction houses would not try to reduce or even eliminate the possibility of lost bids since it is profitable for them to do so.[10]

On the other hand, Eric Rasmusen (2006) assumes that uninformed bidders ini-

[9] This principle, in its more modern form, is usually referred as the Occam's razor, though there is no direct evidence that William of Occam himself ever wrote it down.

[10] One possible explanation is the auction house wants to keep the bidders from leaving the auction house. However, it is not clear why bidders would leave because their bids are not less likely to be lost. See Dan Ariely, Axel Ockenfels and Alvin E. Roth (2005) for other reasons they suggest. Also note that this "collusive bidding" equilibrium supported by the non-transmission of the last minute bid is just one of the (possibly many) equilibria.

tially do not know their private valuation and has to incur a cost to find out. When facing an informed bidder who knows his own value, the uninformed bidder might be able to "save" the cost of finding out if the informed bidder has a high valuation. This timing of finding out one's value induces last minute bidding behavior. Although the discovery of one's value shares similar intuition as in the last minute revision in our paper, the setting is rather specific and special.[11] In contrast, the repetition assumption is rather natural and general.

The rest of the paper is consisted as following. Section I describes a simple example to illustrate the main results. Section II describes the repeated eBay auction format, while section III characterizes the last minute bidding equilibrium, and hence, provides an explanation for last minute bidding. Section IV discusses other auction formats showing that we cannot easily explain last minute bidding without repeated auctions, and section V concludes.

I. An Example: $v_1, v_2, v_3 \sim$ iid uniform $[0, 1]$

Before we go to the general setting, we illustrate our main results with a simple example. Consider the case where there are three bidders with unit demand bidding on two identical items sold sequentially by two sellers. Bidders have valuations v_1, v_2 and v_3, independently drawn from the distribution uniform $[0, 1]$, while sellers do not value the items at all.

The first item is auctioned off through a second price auction where there are multiple rounds of bid submission. After each round of bid submission, the high bidder is revealed, as well as the current price, which is the second highest bid (up to now). Then a next round of bid submission is allowed, save new bids have to be strictly

[11] Here, it is crucial that the uninformed bidder faces an informed one. If both bidders are uncertain about their values, to get last minute bidding behavior, this model requires bidders only get imperfect signals about their private values, which is actually not private values, but (independent) "affiliated value" in Paul R. Milgrom and Robert J. Weber (1982) general model. See Eric Rasmusen (2005).

higher than the current price. At the end of the last round, the bidder who submits the highest bid wins the item and pays the then current price. After the first item is sold, the procedure is repeated for the second one as well.

In the second auction, only two bidders are left. In that case, bidders would bid their valuation during a certain round of bidding since not doing so is a dominated strategy, and hence, the winner would pay a price equal to the other bidder's valuation.[12] Considering only increasing equilibrium, the bidder with the highest valuation $V_{(1)}$ would win the first item, leaving $V_{(2)}$ and $V_{(3)}$ to fight over the remaining item. Hence, the price of the second item would be $V_{(3)}$.

Now we consider the first auction. Without loss of generality, consider $v_i = v_1$. Without further information, bidder v_1 would not be willing to pay a price higher than what she expects to pay in the second auction (when she is tied to win), which is

$$E[V_{(3)}|V_{(1)} = v_1, V_{(2)} = v_1] = E[V_{(3)}|V_{(2)} = v_1] \text{ (by independence)}$$

or the expectation of the third highest valuation conditional on being the (tied) winner of the first auction.

We now calculate $E[V_{(3)}|V_{(2)} = v_1]$ by first obtaining $g(v_{(3)}|V_{(2)} = v)$. Since

$$\begin{aligned} G(y) &= Pr(V_{(3)} < y|V_{(2)} = v) \\ &= \frac{2\int_0^y dv_2 \int_v^1 dv_3}{2\int_0^v dv_2 \int_v^1 dv_3} = \frac{y}{v} \end{aligned}$$

we have

$$g(y|V_{(2)} = v) = \frac{1}{v} \text{ for } 0 \leq y \leq v.$$

Hence,

$$E[V_{(3)}|V_{(2)} = v_1] = \int_0^v \frac{y}{v_1} dy = \frac{v_1}{2}$$

On the other hand, seeing bidder 2 bid b in earlier rounds, and (in equilibrium) correctly inferring that she has $V_2 \geq b$ for b yields a revision of $G(y|v)$. If $b \leq v$, the

[12] They would even bid in the first round if there is, say, a minimum increment that might prevent them from bidding in later rounds.

revision is:

$$\begin{aligned} G(y|v) &= Pr(V_{(3)} < y | V_{(2)} = v_1, V_2 \geq b) \\ &= \frac{\int_b^y dv_2 \int_v^1 dv_3 + \int_0^y dv_3 \int_v^1 dv_2}{\int_b^v dv_2 \int_v^1 dv_3 + \int_0^v dv_3 \int_v^1 dv_2} \\ &= \begin{cases} \frac{2y-b}{2v-b}, \text{ if } b \leq y \leq v, \\ \frac{y}{2v-b}, \text{ if } 0 \leq y \leq b. \end{cases} \end{aligned}$$

Hence, we have

$$g(y|v) = \begin{cases} \frac{2}{2v-b}, \text{ if } b \leq y \leq v, \\ \frac{y}{2v-b}, \text{ if } 0 \leq y \leq b \leq v. \end{cases}$$

Therefore, for $b \leq v_1$,[13]

$$\begin{aligned} & E[V_{(3)} | V_{(2)} = v_1, V_2 \geq b] \\ &= \int_0^b \frac{y}{2v_1 - b} dy + \int_b^{v_1} \frac{2y}{2v_1 - b} dy \\ &= \frac{v_1 - \frac{b^2}{2v_1}}{2 - \frac{b}{v_1}} \geq \frac{v_1}{2}. \end{aligned}$$

In other words, when a bidder bids early, other bidders who have a valuation higher than the early bid would have a new $G(y|\cdot)$ first order stochastically dominating the old one, and hence, have a higher $E(y|\cdot)$. Therefore, bidding in earlier rounds would raise high valuation opponents' conditional expectation of next auction's price, and hence, raise their bids in this auction, which would hurt the bidder if she wins. Thus, bidders would prefer *not* to bid in early rounds, and only bid in the last round. In fact, bidders would delay all bidding until the very last round to avoid pushing other's bids upward in later rounds. In other words, last minute bidding occurs in the first auction.

This example is not special, since we can easily generalize the above result to N bidders and/or to distributions other than uniform, under independent private value

[13] Note that when $b \geq v_1$, $G(y|v_1)$ is unchanged. Hence, when $b \geq v_1$, $E[V_{(3)}|\cdot]$ remains unchanged as well.

(IPV). Furthermore, going beyond independent values is also feasible, as shown in the general model.

II. The eBay Auction

We now present the basic settings of the general model.

A. *Basic Settings*

We adopt the symmetric affiliated private value setting as in Milgrom and Weber (1982, 2000) where bidders have unit demand.[14] Specifically, we assume that there are N bidders, M sellers where $N > M$. There is no discounting and all players are risk neutral. The entire sequential auction is divided into M stages where one seller auctions off one item in each stage. The sellers value the items at $v_0 = 0$. Bidders only want one item and have valuations v_i, which are called their types. Types v_i are private and drawn from the symmetric[15] joint distribution function $F(v_1, v_2, \cdots, v_N)$ and pdf $f(v_1, v_2, \cdots, v_N)$, with support $V = [\underline{v}, \overline{v}]$. For cases with a minimum increment s, we assume $\underline{v} \geq s > 0$ to avoid problems for the lowest type. Types are *affiliated*.[16] The order statistics of $\{v_1, v_2, \cdots, v_N\}$ are $\{V_{(1)}, V_{(2)}, \cdots, V_{(N)}\}$ where

$$V_{(1)} \geq V_{(2)} \geq \cdots \geq V_{(N)}.$$

To simplify things, we focus on symmetric sequential (perfect Bayesian) equilib-

[14]Since last minute bidding can be easily explained in a common value setting, we do not embrace the most general affiliated value model of Milgrom and Weber (1982, 2000).

[15]i.e. for all $1 \leq i, j \leq N$, $F(v_1, \cdots, \mathbf{v_i}, \cdots, \mathbf{v_j}, \cdots, v_N) = F(v_1, \cdots, \mathbf{v_j}, \cdots, \mathbf{v_i}, \cdots, v_N)$.

[16]See Milgrom and Weber (1982) for the formal definition and properties of affiliation. In particular, for vectors z and z', let $z \vee z'$ be the component-wise maximum, and $z \wedge z'$ be the component-wise minimum. Then, random variables $\mathbf{X}$ with joint pdf $f(\mathbf{X})$ is *affiliated* if, for all z and z',

$$f(z \vee z')f(z \wedge z') \geq f(z)f(z').$$

rium. Also, we do not consider (weakly) dominated strategies,[17] and assume that there is no waiting cost.[18]

B. *The eBay Auction Rules*

The eBay auction has several "non-standard" features that makes it different from English auctions described in the literature.

First of all, in eBay auctions, we observe out-cry bids instead of drop-out prices. This is contrary to what is done in the literature. In auction theory, English auctions are typically modeled as *button auctions* (with or without reentry), in which an auctioneer announces prices, and the bidders indicate whether they stay in or not by "pressing on a button" indicating their participation. Bidders decide which price to drop out and let go of the button at that price. Drop-outs are observed immediately, and the auction ends when all but one bidder have dropped out. The remaining bidder wins the item and pays the *current price*, which equals to the price when its last rival dropped out. In the *irrevocable button auction*, as in Milgrom and Weber (1982), the drop-out decision is irrevocable so that bidders cannot return once letting go of their button, while in the *button auction with reentry*, as in Sergei Izmalkov (2003), bidders are allowed to return by pressing on the button again. In both settings, dropping out at one's valuation is a weakly dominant strategy if bidders are assumed to have private values.[19]

[17]Note that asymmetric equilibria arise even in the single unit case if weakly dominated strategies such as bidding zero is allowed. In particular, having one bidder bidding very high and all others bidding zero is an equilibrium for a sealed-bid second price auction.

[18]The "no waiting cost" assumption might not hold in internet auctions, but is always assumed in the auction literature. The key thing needed for our results is that the waiting cost for being available *exactly* at the last minute is lower than the gains from "last minute revisions." Hence, we could approximate this is by having an automated sniper program bid on your behalf if you are not available.

[19]As discussed in Izmalkov (2003), reentry is introduced to restore efficiency in an interdependent value setting. All existing efficient equilibrium without reentry can be easily reproduced when entry is allowed. Hence, in the private setting considered in this section, the equilibrium of the irrevocable button auction can be reproduced even *with* reentry.

However, neither way described above is suitable to model the eBay auction since in the eBay auction we only observe bidders' out-cry bids instead of "drop-outs."[20] Moreover, the interesting phenomenon of "last minute bidding" is specifically the case where bidders hardly ever call out bids save at the last minute, which is equivalent to 'almost always staying out" in these button auctions. Therefore, we need to model the eBay auction in a different way,[21] as a type of *out-cry auction*, in which out-cry bids instead of drop-outs are observed.

Secondly, eBay auctions utilizes a *proxy bidding system* where bidders can submit their reservation prices and let the proxy system bid for them. Bidding on the bidders behalf, the proxy bidding system will only bid up to the amount which is necessary. In particular, the system would only overbid the current rival bid (current highest opponent bid) by at most the minimum increment s. When your bid is the standing bid and another bidder overbids you, the proxy bidding system will revise your bid to counter it unless the other bidder's bid exceeds your reservation price. Although agent-bidding is sometimes observed in physical auction houses as well, the auction literature has mainly ignored such a possibility. There are two main effects of the proxy bidding system. On the one hand, it eliminates the possibility of jump-bidding since the current price only increases up to the rival bid, and hence, eliminates the possible jump-bid equilibria.[22] On the other hand, the proxy bidding system simplifies the bidding process since it automatically makes counter bids for you immediately when you are overbid by another bidder. Besides these two main effects, the proxy bid-

[20] This information issue matters in a repeated setting since we learn bidders' valuations by observing the price they drop out. In particular, Milgrom and Weber (2000) showed that an M times repeated irrevocable button auction where drop-outs are observed would yield exactly the same (ex post) prices in each single-unit irrevocable button auction since we observe the valuation of the $(M+1)$th highest bidder when that bidder drops out. However, such an equilibrium can not exist if "drop-outs" are not observed.

[21] similar to Ockenfels and Roth (2006), and probably first analyzed by Robert Wilson (1975).

[22] Jump bidding can play a role in signalling bidder valuation, as in Avery (1998). However, such discussion is beyond the scope of this paper.

ding system, when joined with other features, can dramatically change the rules of an auction as discussed below.

To summarize, the eBay auction has the following bidding procedure (in each auction stage):

Assumption 1. (Out-Cry Auction with Proxy Bidding)

1. *Submitting Proxy Bids: Bidding starts from the reserve price $r = v_0$ at time $t = 0$. Bidders can submit a "proxy bid" any time $t \in [0, T]$. If no bid is submitted in $[0, T]$, the auction ends without a sale. Bidder i's* proxy bid *at time t is recorded as $b_i(t)$. However, this proxy bid is not revealed immediately. The (hidden) bidding record of proxy bids at time $\tilde{t}$ is $h_{\tilde{t}} = \left\{ b_i(t) \right\}_{t \in [0, \tilde{t})}$*

2. *High Bidder and its Standing Bid: For any time $t \in [0, T]$, the* high bidder, *denoted by bidder k, is the bidder who submitted the highest proxy bid*

$$b_k(t_1) = \max_{b_i(\tilde{t}) \in h_t} b_i(\tilde{t}), \quad t_1 = \arg \max_{\tilde{t} \in [0,t)} \left\{ b_i(\tilde{t}) \middle| b_i(\tilde{t}) \in h_t \right\}.$$

The second-highest bidder, denote by bidder j, is the bidder who submitted the second-highest proxy bid

$$b_j(t_2) = \max_{b_i(\tilde{t}) \in h_t \setminus \{b_k(t_1)\}} b_i(\tilde{t}), \quad t_2 = \arg \max_{\tilde{t} \in [0,t)} \left\{ b_i(\tilde{t}) \middle| b_i(\tilde{t}) \in h_t \setminus \{b_k(t_1)\} \right\}.$$

If there are two or more bidders who submitted their proxy bids, the standing bid *or "current price" S_t at any time $t \in [0, T]$ is $b_j(t_2)$, which is the highest proxy bid that is submitted by a different bidder.*[23] *If there is only one active bidder,*

[23] In practice, eBay actually has a standing bid of

$$S_t = \min \left\{ b_j(t_2) + s, b_k(\bar{t}) \right\}$$

which is the highest proxy bid that is submitted by a different bidder plus an increment unless the difference between the two proxy bids is smaller than the increment. However, this creates a small possibility that the eBay auction becomes a first price auction (when the two highest bidders are close), which makes bidding one's valuation no longer a dominant strategy even in the single auction case. In fact, with only one auction, *the reserve price r. If two bidders submitted the same highest proxy bid, the bidder who submitted their proxy bid(s) earlier becomes the high bidder, and the standing bid equals to the tied highest proxy bid. The standing bid S_t is common knowledge immediately after time t.*[24]

3. *Ascending Proxy Bids: Later proxy bids have to be higher than the bidder's last submitted proxy bid*[25] *and higher than the standing bid by a minimum increment s.*[26] *In other words, if $b_i(\hat{t})$ is the last proxy bid submitted by bidder i, then its new proxy bid $b_i(t)$ must satisfy*

$$b_i(t) > b_i(\hat{t}), \qquad \textit{and} \qquad b_i(t) > S_t + s.$$

Thirdly, the eBay auction has a *fixed-time ending rule*, which ends the auction at a pre-set time, commonly known to all bidders, regardless of the bidding behavior during the course of the auction. This is contrary to the button auction which ends when all other bidders drop out, and differs from typical out-cry auctions seen in the

the first order condition now requires the bid function to satisfy: (for $\delta(v_i) = v_i - \beta_1^{-1}(\beta_1(v_i) - s)$)

$$\beta(v) \quad = \quad v - \underbrace{\beta'(v) \cdot \frac{F_1(v|v) - F_1(v - \delta(v)|v)}{f_1(v|v)}}_{\text{shade as if first price}} .$$

[24]Note that once a proxy bid is overbid, it is automatically revealed through change in the standing bid. Hence, proxy bids are usually revealed with a delay, except the winning proxy bid, which is never revealed—not even after the end of the auction.

[25]In other words, a bidder cannot lower its still hidden proxy bid. The only way to lower an existing proxy bid, is to retract it and re-submit a new one. However, bid retractions are extremely rare in practice, and if ever happens, is viewed as bad signals. (Note that if bid retraction is costless, the sellers would have a strong incentive to bid himself, and retract her bid whenever she (accidentally) becomes the high bidder.)

[26]Note that this minimum increment rule does *not* apply on others' (hidden) proxy bid, but only on the standing bid S_t. Hence, it is possible to submit a proxy bid close to the (hidden) proxy bid. This is quite different from the continuous time out-cry auction. Another difference is that the ascending rule here only apply to bids submitted *after* the standing bid, but not those that were submitted *simultaneously*.

field which end when there are no further bids.[27]

In other words, we have:

Assumption 2. (Fixed-Time Ending Rule) *The auction ends at a fixed time T. The high bidder wins the item pays the standing bid S_T. Note that due to the proxy bidding system, bidders may submit their reservation price at time $t = T$. This makes the "last minute" T* as if *a sealed bid second price auction.*

Last but not the least, since there are many independent buyers and sellers participating on eBay, identical items are likely to be sold separately but *repeatedly* on eBay. Repeatedness does not drastically change the rules of the auction, except that the auction house repeats the same single-unit auction M times, each for one item. We denote each single-unit auction as an "auction stage." Repetition pushes the eBay auction away from the single-unit auction framework, and moves towards a multi-unit, market like structure. Also, it makes bidders not only care about their own private values, but also what they might need to pay in subsequent auctions.

To see the influence of repetition, we first define the bid history of the repeated eBay auction:

Assumption 3. (History in the Repeated eBay Auction)

1. *Revelation of Last Minute Bids:*
   We assume that all last minute bids are accepted and ordered from the smallest to the largest. Hence, no last minute bid would be rejected due to the ascending rule.[28]

[27] As shown below, the ending rule which requires no further bids can be called the "going-going-gone" rule, and is a key difference between eBay and Amazon auctions.

[28] Note that if the last minute bids do not come in exactly at the same time, information revelation is actually determined by the the order of these last minute bids. Later bids which are lower then the standing bid (plus an increment) are rejected by the ascending bid rule, and hence, are not revealed. If we assume the order of the last minute bids are random, the revelation is also somewhat random. Alternatively, we might think of endogenously determining the order. If last minute bids are ordered from the smallest to the largest, most

2. *Public Bidding Record of Past Stages:*

   At any stage m, the bidding record of the ended auction at stage $m' < m$ is the sequence of proxy bids (except the winning one)[29]

$$\tilde{h}_{m',T} = \left\{ b_i^{m'}(t) \right\}_{t \in [0,T]} \setminus \left\{ \max_{t \in [0,T]} b_i^{m'}(t) \right\}$$

3. *Public Bidding Record of Current Stage:*

   The bidding record of the current stage m, time $t \in [0, T]$ is denoted by the sequence of standing bids $S_{m,\tau} \in V \cup \{\emptyset\}$, as well as the identity of the bidders $i_\tau \in I_0 = \{0, 1, \cdots, n_{m,t}\}$ who submitted proxy bids at $\tau \in [0, t]$, and can be transformed into a sequence of proxy bids[30]

$$\tilde{h}_{m,t} = \left\{ b_i^m(\tilde{t}) \right\}_{\tilde{t} \in [0,t)} \setminus \left\{ \max_{\tilde{t} \in [0,t)} b_i^m(\tilde{t}) \right\}$$

   where $b_i^m(\tilde{t})$ is bidder i's proxy bid at time $\tilde{t}$ (of stage m).[31]

4. *Bidding History: The entire bidding history up to stage m, time t is defined by*

$$h_{m,t} = \left\{ \tilde{h}_{1,T}, \cdots, \tilde{h}_{m-1,T}, \tilde{h}_{m,t} \right\}$$

last minute bids (except the highest one and those blocked out by the minimum increment requirement) are revealed. If last minute bids are ordered from the largest to the smallest, then only the second-highest last minute bid, which determines the final standing bid, is revealed. Other cases are in between. Information revelation might influence the results. Nevertheless, as shown below, when there are only two items ($M = 2$), all information structures result in the same equilibrium.

[29]When the two highest proxy bids are very close, there is a chance that the highest bid is also revealed (through standing bid $S_T^{m'}$). However, as shown below, this information does not matter in equilibrium when $M = 2$.

[30]Amazon actually reveals these proxy bids immediately after they are outbid by other bidders. eBay initially delayed this to the end of the auction, but then quietly changed to the Amazon way. This is almost identical, since we can almost always derive the outbid proxy bids from the change in the standing bids. However, there might be a subtle difference if the proxy bids are too close. In this paper, we abstract away from these subtleties.

[31]Note that the winning proxy bid is not revealed. Also, bids submitted at the same time t might be rejected due to the ordering of these simultaneous bids and the ascending rule. To avoid this, we can simply assume simultaneous bids are ordered from small to large so no bid is rejected.

Hence, for any stage m, the past history space is

$$H_m = \prod_{\tilde{m}=1}^{m-1} \left(\left[\prod_{\tilde{t} \in [0,T]} \left((V \cup \{\emptyset\}) \otimes I_0 \right)^n \right]_{\tilde{t}} \right)_{\tilde{m}},$$

and the history space of the current stage is

$$H_c = \prod_{\tilde{t} \in [0,T]} \left[\left((V \cup \{\emptyset\}) \otimes I_0 \right)^n \right]_{\tilde{t}},$$

so that the entire history space is $H = H_{M-1} \times H_c$.

Note that the symmetric bidding strategy for a bidder with valuation v is a function

$$\beta : V \times H \Longrightarrow \{\emptyset\} \cup V.$$

In particular, the bidding strategy bidder i with valuation v_i at time t of stage m (after history $h_{m,t}$) is

$$\beta_i(v_i, h_{m,t}) = \begin{cases} \emptyset & \text{(not bid)} \\ b_i \in V & \text{(bid } b_i\text{)} \end{cases}$$

Therefore, the strategy space is $V \times H \times (V \cup \{\emptyset\})$, which is potentially huge.

To summarize, an *eBay auction* is defined to be an *out-cry auction* with *proxy bidding* and a *fixed-time ending rule*. If there are more than one item sold, the same eBay auction is conducted *repeatedly* using the same rules. The combination of these non-standard features can produce unexpected results. For example, the proxy bidding system at the fixed ending-time works like a one-shot *sealed bid second price* auction. This makes the eBay auction more closer to a sealed bid second price format, and combining with the last minute bidding phenomenon, motivated Bajari and Hortacsu (2003) to analyze eBay coin auctions as if they were actually sealed bid second price auctions.

We first focus on the eBay auction and address the overall effect of all of these non-standard features, and discuss other possible formats and the effect of individual features in the discussion section.

C. *The "Puzzle" of Last Minute Bidding*

The last stage of an eBay auction is simply a single-unit auction. In this environment, we can easily find the unique equilibrium by (weak) dominance. To do this, first observe that any bidding strategy that does not "bid up to one's valuation," i.e. the resulting bidding sequence does not end with a final bid between $v_i - s$ and v_i, is a dominated strategy. This is because one can modify that strategy by adding another bid of one's valuation to the end of that sequence. Moreover, given that everybody else will "bid up to their valuation," a bidder will find bidding her valuation at time zero is (weakly) dominant since she might be "blocked out" from bidding by the minimum increment rule. Therefore, we have that in the last stage of the eBay auction, the unique equilibrium is characterized by bidders submitting proxy bids $b_i(0) = v_i$ at $t = 0$, and cease to bid after the initial bids are submitted. Hence, the unique equilibrium of eBay auction's final stage is the same as that in the (irrevocable) button auction and is efficient.

Note that in this equilibrium, all bidders submit a proxy bid equal to their valuations at the time 0 and the ending rule has no bite.

Also note that when there is only one auction ($M = 1$), all bidders submit proxy bids equal to their valuation *at time* 0, not at the last minute (time T) observed in practice. This is why "sniping," or last minute bidding, is not explained by (and even contradicts) standard (single-unit) auction literature. Specifically, we have the following "puzzle" of last minute bidding:

Proposition 1. (eBay Auction)
Assume that there is only one item sold under the eBay auction rule. The unique equilibrium is characterized by bidders submitting proxy bids $b_i(0) = v_i$ at $t = 0$, and cease to bid after the initial bids are submitted. Hence, the unique equilibrium of the (single-unit) eBay auction is the same as that in the (irrevocable) button auction and is efficient.

Proof. See Appendix.

Hence, in order to explain last minute bidding, Ockenfels and Roth (2006) add an exogenous probability p that the last minute bid submitted at time T is not successfully transmitted due to internet congestion or disconnection. They construct a "collusive" equilibrium in which bidder only bid at the last minute hoping for the small probability of gaining a lot when others' bids are lost. However, it is difficult to imagine how numerous online bidders who never knew each other could jointly collude. What is more, such equilibrium relies heavily on appropriate parameters such as the distribution of values, the probability p and the number of bidders. In fact, Ockenfels and Roth (2006) only gave an example where there are only two bidders with identical values as proof for the *possibility* of such collusive equilibrium, instead of specifying exact conditions for it to happen.[32]

We take an alternative approach, and consider the situation where many identical items are sold in the same (eBay) auction house, but each auction ending at a different time. This adds repetition to the model.

III. Equilibrium of the Repeated Ebay Auction

Note that the eBay auction has the sealed-bid second price auction as a special cases when $T = 0$. In other words, the (repeated) *eBay auction with* $T = 0$ is equivalent to a (repeated) *sealed-bid second price auction*. We first consider this special cases as a benchmark.

[32]Ariely et al. (2005) has another numerical example, but still not a general result. Moreover, Ariely et al. (2005) show that subjects still bid at the last minute even in sessions where the probability p was zero, contrary to their own "collusive" theory. Hence, they explore other non-equilibrium explanations such as best responding to incremental bidders.

A. *The Repeated Sealed-bid Second Price Auction*

When $T = 0$, which eliminates all but the last round of bidding, the eBay auction reduces to a sequence of sealed-bid second price auctions. The equilibrium of this auction can be found through backward induction. In the last stage, as shown above, bidding one's valuation (at $T = 0$) is the (weakly) dominant strategy. Moving back to the early stage, in this second price setting, bidders would bid their expected prices of the next item, conditional on themselves being the tied winner for this one, since this is the "willingness-to- pay" for this auction stage. Hence, we have:

Theorem 1. (APV, Any Information Structure, 2 Stages)
Assume that two items are sold under the repeated sealed-bid second price auction, i.e. the eBay auction rule with $T = 0$, and $s = 0$. Then, no matter how many last minute bids are revealed after the auctions (arbitrary on information revelation assumption 1.), a symmetric equilibrium is determined by bidding $\beta_1(v_i) = E\Big[V_{(3)}\Big|V_{(1)} = v_i, V_{(2)} = v_i\Big]$ in stage 1, and $\beta_2(v_i) = v_i$ in stage 2. Moreover, expected revenue is weakly higher in the last stage, which equals to $E\Big[V_{(3)}\Big]$. In particular, when valuations are independent, we have revenue equivalence such that stage 1 and stage 2 yield the same expected revenue, and total revenue equals to that of a multi-unit uniform price (Vickrey) auction selling all items simultaneously.

Proof. See Appendix.

Now we consider the general case where T is positive and $M = 2$, in which two identical items are sold using the repeated eBay auction rule. First, we show there exists a last minute bidding equilibrium where everyone jumps in and bids at the last minute. Then, we can consider uniqueness of such equilibrium.

B. *Existence of Last Minute Bidding Equilibrium*

To get existence, we need several lemmas about affiliation and information updating. First, note that a bidding strategy without a last minute bidding feature is weakly dominated by the bidding strategy which is almost identical except "adding" a last minute bidding clause, and if a bidder already bid her valuation in some early stage, she would not bid at the last minute even with such a clause. Another thing to note is that revealing information cannot revise the expected prices lower in these eBay open auctions.[33] To get this result, we need properties about affiliation, which is mainly covered in Milgrom and Weber (1982), except the self affiliation property stated below.

To summarize, we have the following three properties of affiliation:

Lemma 1. *Suppose $v_1, v_2, \cdots, v_N$ are affiliated, and we are in the repeated eBay auction game, then*

1. *(Self-affiliation) For $Y = v_1$, $Y, v_1, v_2, \cdots, v_N$ are affiliated.*
2. *(Last minute bidding clause) For $M = 2$ and $s = 0$, a bidding strategy $\beta(v_i, h_{m,t})$ such that $\beta_1(v_i, h_{1,T}) = \emptyset$ is weakly dominated by*

$$\begin{aligned} \tilde{\beta}(v_i, h_{m,t}) &= \beta(v_i, h_{m,t}), \textit{ for all } v_i \textit{ and } (m,t) \neq (1,T). \\ &= E\big[V_{(3)} \big| V_{(1)} = v_i, V_{(2)} = v_i, h_{1,T}\big], \textit{ if } (m,t) = (1,T). \end{aligned}$$

3. *(Belief updating) After any history $h_{m,t}$ of the sequential eBay auction where $V_1, \cdots, V_n$ are still affiliated, for $a > \underline{v}$,*

$$\begin{aligned} &E\Big[\min\{V_{(M+1)} + s, V_i\}\Big| V_{(m)} = v_i, V_{(m+1)} = v_i, V_i = v_i, a < V_j < \overline{v}\Big] \\ &\geq E\Big[\min\{V_{(M+1)} + s, V_i\}\Big| V_{(m)} = v_i, V_{(m+1)} = v_i, V_i = v_i, \underline{v} < V_j < \overline{v}\Big]. \end{aligned}$$

Proof. See Appendix.

Now we show existence by constructing a *last minute bidding equilibrium* where all bidders wait and jump in at the last minute, $t = T$, bidding the *repeated sealed-bid*

[33] Note that Milgrom and Weber (1982) provided a similar result for the seller in the general affiliated value (AV) settings, stated as **Theorem 8**: In the second price auctions, publicly revealing the seller's information raises revenue (weakly).

second price auction equilibrium characterized in the previous section, according the players' ex ante information in the beginning of this stage.[34]

Theorem 2. (Existence of the Last Minute Bidding Equilibrium)
Assume that there are two items sold under the repeated eBay auction rule, and $s \geq 0$*. Then, there exists a (symmetric)* last minute bidding equilibrium *where*

$$\beta_1 = \begin{cases} \emptyset & t < T \\ E\big[V_{(3)}\big|V_{(1)} = v_i, V_{(2)} = v_i\big] & t = T \end{cases}$$

$$\beta_2 = \begin{cases} v_i & t = 0 \\ \emptyset & t > 0. \end{cases}$$

and the out of equilibrium beliefs are

1. *If at some time t,* S_t *moves to become* $S_0 = a > \underline{v}$*, and at the same time bidder* j_1 *replaces bidder* j_2 *as the high bidder, then infer that* $\overline{v} \geq v_{j_1}, v_{j_2} \geq S_0 > \underline{v}$*.*
2. *If at some time t,* $S_t = S_0 \geq 0$*, but* $S_0 \leq \underline{v}$*, then infer that* $\overline{v} \geq v_j \geq a > \underline{v}$ *for some arbitrary given* $a > \underline{v}$*, for all* v_j *that have submitted a bid.*

Proof. See Appendix.

Note that if $s > 0$, this equilibrium might not sustain if the minimum increment s is charged on simultaneous bids since this would create a "first price" situation when valuations are close.[35]

[34] Since there is no information update, the interim information $h_{1,T}$, $h_{2,T}$ is the same as the ex ante information $h_{1,0}$, $h_{2,0}$.

[35] When simultaneous bids are charged with the minimum increment, and hence, the winning price being $\min\{v_{(3)} + s, v_i\}$, bidders have to consider the possibility of falling into the "first price auction" range where $v_{(1)} - v_{(2)} < s$. Hence, the first order condition becomes

$$\beta_1(v_i) = E\big[\min\{Y_2 + s, V_i\}\big|Y_1 = v_i, V_i = v_i\big] - \beta_1'(v_i) \cdot \frac{F_1(v_i|v_i) - F_1(v_i - \delta(v_i)|v_i)}{f_1(v_i|v_i)}$$

where $\delta(v_i) = v_i - \beta_1^{-1}(\beta_1(v_i) - s)$. In other words, when there is a chance of determining the winning price (when the auction is close), bidders shade their bids from their conditional expectations to account for this "first price" possibility.

This is because the ascending rule applies only to "earlier" or "later" proxy bids, so these simultaneous bids would not be blocked out. Moreover, one deviation would *not* crowd others out since the standing bid is only updated to be the reserve price $r = v_0 = 0$, and the lowest type is $\underline{v} \geq s > 0$.

C. *Uniqueness of the Last Minute Bidding Equilibrium*

Now we attempt to argue for the uniqueness of this last minute bidding equilibrium. To do this, we eliminate bidding strategies that are weakly dominated. Recall that focusing on symmetric equilibrium already rules out asymmetric equilibria where one bidder bids very high and others bid zero, which rely on weakly dominated strategies.

We can now state our uniqueness theorem:

Theorem 3. (Uniqueness of the Last Minute Bidding Equilibrium)
The last minute bidding equilibrium is the unique symmetric equilibrium if we consider only undominated and monotonic strategies for the repeated eBay auction rule. In other words, people simply play the last minute bidding equilibrium strategy where everybody jumps in at the last minute except in the last stage, and hence, bid according to their ex ante information, obtaining the repeated sealed-bid second price auction outcome.

Proof. See Appendix.

Intuitively, since bidding early reveals information and increases your opponents' conditional expectation, and hence, increases their last minute bids, you would rather withhold bidding until the last minute.

According to the theorem, last minute bidding can be rationalized even in the private value case, and we have shown that with repeated auctions, bidders' reservation prices in the first period are actually correlated across bidders since the conditional expectations of $v_{(M+1)}$, which are their ex ante reservation prices in the early stages, depends on not only one's type but others' types as well.

Note that there are two interesting empirical implications of this theorem. First of all, if the auctions observed are repeated, private values(PV) and common values(CV) would be difficult to distinguish empirically, since even if the true valuations *were* private values, the bids would still depend on other bidders' information in early auctions. Unfortunately, this is typically the case since similar items are auctioned off over and over across time, and in fact, we need to observe similar auctions performed repeatedly to identify our empirical estimates.

Moreover, though in principle we should find no winner's curse in private value auctions, there might still be winner's curse in *repeated* private value auctions. To be precise, in the theoretical analysis above, bidders avoid the winner's curse since all of the expectations were conditional on $V_i = V_{(m+1)} > V_{(M+1)}$. However, if people do not condition on that, they would still suffer the winner's curse.

D. *Revenue Ranking and Efficiency*

Since the symmetric equilibrium we consider assumes monotonic strategies, the winners of stage $1, \cdots, M$ have valuations $v_{(1)}, \cdots, v_{(M)}$, respectively. Therefore, the outcome is efficient. Moreover, sellers revenue differ between stages. In fact, from the proof of Theorem 1, we have shown that

Theorem 4. (Revenue Ranking; APV, $M = 2$)
Suppose there are two items sold ($M = 2$) under the eBay auction (or sealed-bid second price auction) rule, and $s = 0$. Then, the expected revenue of each stage is $E\Big[\beta_1(V_{(2)})\Big]$ and $E\Big[V_{(3)}\Big]$ where $E\Big[\beta_1(V_{(2)})\Big] \leq E\Big[V_{(3)}\Big]$, and hence, sellers' total expected revenue is lesser or equal to $2 \cdot E\Big[V_{(3)}\Big]$, which is what they would earn if they combined the items and held a joint sealed-bid auction where the winning price is the highest loser's bid.

Thus, with affiliation, last minute bidding does not hurt efficiency, but might leave room for sellers to improve in terms of revenue. In fact, if the first seller could join with the second seller to host a multi-unit second auction in which the first and second highest bidder win an item and pay the third highest price, the first seller's expected revenue would increase, while the second seller would remain indifferent.

Although revenue equivalence generally breaks down in the APV case,[36] with independence, we still have revenue equivalence, as shown in the proof of Theorem 1. In other words, under independence, the repeated eBay auction yields the same expected revenue for each stage, and sellers are indifferent between selling in different stages. Therefore, under independent private value, last minute bidding is not "bad" in the sense of both efficiency and revenue.[37] We restate it as

Corollary 5. (Revenue Equivalence; IPV, $M = 2$)
Suppose v_i are independent, there are two items sold ($M = 2$) under the eBay auction (or sealed-bid second price auction) rule, and $s = 0$. Then, the expected revenue of each stage is $E\left[V_{(3)}\right]$, and the sellers' total expected revenue is $2 \cdot E\left[V_{(3)}\right]$.

This result is a special case of Weber (1983) proved the general revenue equivalence theorem for the IPV case.[38]

[36] Milgrom and Weber (1982) rank different auction rules according to their expected revenue in the general setting. They argue that irrevocable button auctions are widely used due to its higher expected revenue, compared to the sealed-bid first price or second price auctions.

[37] The revenue ranking for more than two items under affiliated private value is partially discussed in Joseph Tao-yi Wang (2006). In particular, the sealed-bid second price auction (when the highest two bids are revealed) yields lower revenue for earlier auctions than later ones. A similar last minute bidding equilibrium would exist in the eBay auction as well, but uniqueness may not hold.

[38] Specifically, Weber (1983) has:

Proposition. (Revenue Equivalence; IPV; Weber)
Suppose v_i are independent. For any auction rule that assumes that in equilibrium, the M highest types win the auction for sure, and the lowest type $\underline{v}$ expects to get 0, the total seller(s) expected revenue is $M \cdot E\left[V_{(M+1)}\right]$.

IV. Discussion: Other Auction Formats

After presenting the main results of the eBay auction, we now turn to discuss other possible auction formats and demonstrate why last minute bidding behavior cannot be easily explained. As discussed earlier, the eBay auction is an out-cry auction with two special features: the fixed-time ending rule *and* the proxy-bidding system. We may consider other types of out-cry auctions that have only one (or neither) of the these special features.

Without any special features, we have the *simple out-cry auction*. This is typically the way English auctions are carried out in the field, though the auction literature usually models it as a button auction (with or without re-entry).[39] Adding a fixed ending time to the simple out-cry auction, we have a *fixed-ending out-cry auction*. On the other hand, if we do not have a fixed ending rule, but add a proxy-bidding auction, we obtain an *Amazon auction*, named after the online auction house which uses this format. We may summarize the four different variants of the out-cry auction as follows:

		Proxy-bidding System	
		Yes	No
Fixed-time	Yes	eBay auction	Fixed-ending out-cry auction
Ending Rule	No	Amazon auction	Simple out-cry auction

We now proceed to discuss these formats.

A. *The Simple Out-cry Auction*

As discussed earlier, there are two ways to model the English auction in the auction literature. One is the *irrevocable button auction* analyzed by Milgrom and Weber (1982), and the other is the *button auction with reentry* analyzed by Izmalkov (2003).

[39]Indeed, we show that there exist a *button auction equilibrium* in the simple out-cry auction which resembles the equilibrium of a button auction.

In both cases, the auctioneer announces prices, and the bidders indicate whether they stay in or not by pressing on a button indicating their participation.

However, as a by product of discussing eBay auctions, we model the English auction in a third way, as an *out-cry auction* with the following bidding rules:

Assumption 4. (Out-cry Bidding Procedure)

1. *Submitting Bids: Bidding starts from the reserve price* $r = v_0$ *at time* $t = 0$*. Bidders can bid at any time* $t \in [0, T]$*. If no bid is submitted in* $[0, T]$*, the auction ends without a sale. Bidder* i*'s bid at time* t *is recorded as* $b_i(t)$*. This bidding record is available immediately after time* t*. Hence, the bidding record at time* $\tilde{t}$ *is* $h_{\tilde{t}} = \left\{b_i(t)\right\}_{t\in[0,\tilde{t})} \in H$ *where* $H = \prod_{t\in T}(V \cup \{\emptyset\})^n$ *is the history space.*

2. *Bidding Sequence: Bids are ordered according to its submission time* t*. If two or more bids enter at the same instance* t*, they are randomly ordered. i.e. the bidding sequence is* $b_1 = b_{i_1}(t_1)$*,* $b_2 = b_{i_2}(t_2)$*,* $\cdots$ *,* $b_n = b_{i_n}(t_n)$*,* $\cdots$*, where* $t_k \leq t_{k+1}$*.*

3. *Ascending Bids: Later bids have to be higher than their predecessors by a minimum increment* s*. In other words, if the bidding sequence is* b_1*,* b_2*,* $\cdots$ *,* b_n*,* $\cdots$ *, then* $b_{k+1} \geq b_k + s$ *for all* $k \in \mathbf{N}$*.*[40] *We denote the last and highest bid (at time* t*) as the* standing bid *or* current price S_t*.*

4. *Symmetric Bidding Strategies: The bidding strategy for a bidder with valuation* v *is a function*

$$\beta : V \times H \Longrightarrow \{\emptyset\} \cup V$$

[40] If two bids are submitted at the same instance t, the (randomly selected) latter bid $b_{i_{k+1}}(t_{k+1})$ would be rejected and not recorded if it does not satisfy this minimum increment requirement. However, this is fine since the bidder can always resubmit another bid at $t' > t$ if she has valuation $v_{i_{k+1}} \geq S_t + s$. Only if there is a fixed last minute to submit bids or if the minimum increment is larger than the difference between $v_{i_{k+1}}$ and S_t would this be an issue. Nevertheless, if $|v_{i_{k+1}} - S_t| < s$, the potential efficiency loss is at most s, and vanishes as $s \to 0$.

The bidding strategy bidder i with valuation v_i at time t (after history h_t) is

$$\beta_i(v_i, h_t) = \begin{cases} \emptyset & \textit{(not bid)} \\ b_i \in V & \textit{(bid } b_i\textit{)} \end{cases}$$

Just as in reality, there is an auctioneer soliciting bids several times before announcing that the item is sold, we have the *going-going-gone ending rule*:

Assumption 5. (going-going-gone ending rule) *The auction ends if no further bids came in during the "going-going-gone" period $\left(t_n, t_n + \overline{T}\right]$ after the last bid was submitted at t_n. The high bidder wins the item and pays the standing bid S_{t_n}. If $t_n + \overline{T} > T$, then the auction is* automatically extended *to time $T' = \left(t_n + \overline{T}\right)$.*[41] *The bidding history of this ended auction is the finite*[42] *bidding sequence $h = \left\{b_{i_k}(t_k)\right\}_{k=1}^{n}$ where $b_k(\cdot)$ is bidder i_k's bid at time t_k.*

A *simple out-cry auction* is an *out-cry auction* with a *going-going-gone ending rule*.[43] When there is only one auction conducted, i.e. $M = 1$, we may characterize the equilibrium of the simple out-cry auction by the following lemma:

Lemma 2. *Assume that there is only one item sold ($M = 1$) under the simple out-cry auction rule. If it is common knowledge that $s < v_{(1)} - v_{(2)}$, an equilibrium is characterized by a finite bidding sequence $\{b_1, \cdots, b_n\}$ with the bidder with valuation $v_{(1)}$ bidding the last bid $b_n \in \left((v_{(2)} - s), (v_{(2)} + s)\right]$, and all other bidders cease to bid during the* going-going-gone *period $\left(t_n, t_n + \overline{T}\right)$. Hence, the outcome is efficient, and as $s \to 0$, the winning price converge to the winning price in the (irrevocable) button auction.*

Proof. See Appendix.

[41] Actually, automatic extension is not necessary if T is large enough, just as in practice the auctioneer usually has enough time to solicit for new bids. What is crucial is the "going-going-gone" ending rule.

[42] Since we have a minimum increment s, the sequence of bids is finite as long as the winning bid is not infinity.

[43] This is in contrast to the *fixed-ending out-cry auction* discussed in the next section.

In fact, this result may be the very reason previous literature modeled English auctions directly *as* button auctions, even though they are actually conducted in the field as simple out-cry auctions.

In any case, if the valuations have a smooth pdf, we can actually specify the equilibrium as follows:

Proposition 2. (Simple Out-cry Auction)
Assume that there is only one item sold ($M = 1$) under the simple out-cry auction rule. If f_2, the pdf of $V_{(2)}(= Y_1)$, satisfies

$$\frac{f_2(y+\eta)}{f_2(y)} \leq 4 \textit{ for all } y \in [\underline{v}, \overline{v}] \textit{ and } \eta \in [s, 3s],$$

there exists an **button auction equilibrium (BAE)** *characterized by a series of bids*

$$b_i(t) = s, 2s, \cdots, ks, \cdots, \textit{ at } t = \tilde{T}, 2\tilde{T}, \cdots, k\tilde{T}, \cdots,$$

as long as $v_i \geq ks$, for $\tilde{T} \leq \overline{T}$.
As $s \to 0$, the above assumption becomes less and less binding, and the outcome approximates an irrevocable button auction.

Proof. See Appendix.

Intuitively, the button auction equilibrium produces an outcome that mimics the price clock in a button auction. Hence, there is little to gain by deviating from the equilibrium strategy. However, smoothness is required here so bidders cannot gain by jump bidding and blocking opponents with valuation right above her. This requirement is less and less binding as $s \to 0$, and is trivially satisfied when $s = 0$.

Note that when $M \geq 2$, the equivalence result between the (irrevocable) button auction and the continuous-time out-cry auction may not hold.[44]

[44] This is because the simple out-cry auction does *not* require all other bidders to counter bid, which is equivalent to not dropping out in the (irrevocable) button auction, but just at least *one* counter bid. Hence, two bidders might engage in a bidding war against each other which is unnecessary since one of them could stop

B. *The Fixed-ending Out-cry Auction*

Without the proxy-bidding system, the fixed-time ending rule would make the "last minute" work like a first price auction. Hence, this would make the out-cry auction end with one last round of sealed bid first price auction. Such an auction format would be similar to the Anglo-Dutch auction proposed by Kenneth G. Binmore and Paul Klemperer (2002) for the British 3G telecom auction, except that the last round of sealed bid first price auction is triggered by a fixed ending time. We refer this auction format as the *Fixed-ending Out-cry Auction*, which is literally the eBay auction without proxy-bidding.

If there is only one item sold ($M = 1$), the fixed-ending out-cry auction has a button-auction equilibrium (after re-mapping bidding time into the fixed time period), just as the simple out-cry auction. In other words, we have the following proposition:

Proposition 3. (Fixed-ending Out-cry Auction)
Assume that there is only one item sold ($M = 1$) under the fixed-ending out-cry auction rule. If f_2, the pdf of $V_{(2)}(= Y_1)$, satisfies

$$\frac{f_2(y+\eta)}{f_2(y)} \leq 4 \textit{ for all } y \in [\underline{v}, \overline{v}] \textit{ and } \eta \in [s, 3s],$$

there exists a **button auction equilibrium (BAE)** *characterized by*

$$b_i(t) = s, 2s, \cdots, ks, \cdots, Ks, \textit{ at } t = \tilde{T}, 2\tilde{T}, \cdots, k\tilde{T}, \cdots, K\tilde{T}$$

where $K = \left[\frac{\overline{v}}{s}\right] + 1$, the closest (but higher) integer to $\frac{\overline{v}}{s}$, and $\tilde{T} \leq \frac{T}{K}$, guaranteeing bidding would be "complete" before the fixed ending time. as long as $v_i \geq ks$, for $\tilde{T} \leq \overline{T}$.

counter bidding, wait, and bid on subsequent auctions. However, this creates incentives for high valuation bidders to jump bid, eliminating the chance for multiple opponents to counter bids revealing their relatively high valuation, and hence, deter other bidders from counter bidding. Such jump-bid equilibria rationalizes the imposed game structures in the jump-bid literature such as Avery (1998). However, further discussion is beyond the scope of this paper.

As $s \to 0$, the above assumption becomes less and less binding, and the outcome approximates an irrevocable button auction.

The proof is identical to that in the previous section.

One might speculate that a last minute bidding equilibrium could exist in this auction format where bidders bid as if it were a sealed-bid first price auction. However, to demonstrate the difficulty of explaining last minute bidding with the fixed-time ending alone, it is suffice to show that a button auction equilibrium exists in the fixed-ending out-cry auction. Further consideration is beyond the scope of this paper.

C. *The Amazon Auction*

As shown earlier, the eBay auction has several non-standard features that makes it different from the single out-cry auction considered in the previous section. First, it has a *proxy bidding system* where you can submit your reservation price and let the proxy system bid for you. Moreover, the eBay auction has a *fixed-time ending rule,* and thus, the proxy bidding system at the fixed ending-time works more like a sealed bid *second price* auction.

On the other hand, as reported in Roth and Ockenfels (2002), Amazon.com has its own auction site which conducts auctions similar to eBay, which also has a proxy-bidding system but preserving the "going-going-gone" ending rule of the simple out-cry auction. We refer this auction format, which consists of *proxy bidding* and the *going-going-gone ending rule*, as the *Amazon auction*.

For the case where only one item is sold, or $M = 1$, we can characterize the unique equilibrium of the Amazon auction with the following proposition:

Proposition 4. (Amazon Auction)

Assume that there is one item sold ($M = 1$) under the Amazon auction rule and $s > 0$. Then, the unique equilibrium is characterized by bidders submitting proxy bids $b_i(0) = v_i$ at the beginning of the auction, and cease to bid after the initial bids are *submitted. Hence, the unique equilibrium of the Amazon auction is the same as that in the (irrevocable) button auction and is efficient.*

Note that in this equilibrium, all bidders submit a proxy bid equal to their valuations at the time 0 and the automatic extension rule has no bite. Hence, the proof is identical to that of Proposition 1.

In conclusion, the equilibrium for each auction format is given by Proposition 1 through 4, and we may summarize equilibria under the four different variants of the out-cry auction as follows:

		Proxy-bidding System	
		Yes	No
Fixed-time Ending Rule	Yes	(eBay) Bid at $t = 0$	(FEOC) Button Auction Eq.
	No	(Amazon) Bid at $t = 0$	(SOC) Button Auction Eq.

Moreover, since bidding at time 0 is dominant when there is proxy bidding, last minute bidding is not generated by the fixed-time ending rule per se. Nevertheless, without the fixed-time ending rule, there is no "last minute" to bid on! Therefore, neither proxy bidding, nor a fixed ending time, can induce last minute bidding by itself. In fact, it is only with both of them, and the fact that there are multiple units sold which generates last minute bidding behavior.

V. Conclusion

In this paper, we rationalize last minute bidding by adding (at least) another identical auction. In the repeated eBay auction model, we prove that the last minute bidding equilibrium, in which bidders only bid at the last minute, is the unique symmetric sequential equilibrium in weakly undominated and monotonic strategies. Moreover, bidders do *not* always bid their valuations. In fact, a bidder's maximum willingness to pay for this stage depends on on others' types. This introduces a "common value" component to the private value environment and leads to bid shading in the first auction. Nonetheless, even though high valuation bidders efficiently win the items, under strict affiliation, expected revenue is lower in the first auction than in the last, and the seller of the first item can increase her revenue by running a multi-unit auction together with the second seller. Such revenue gains can be viewed as potential synergies for multi-unit auctions, and the foregone revenue to "auction first" highlights the opportunity cost of coordination failure.

Mathematical Appendix

Proof of Proposition 1. We first show that not bidding up to one's valuation is (weakly) dominated. Then, after eliminating these dominated strategies, it suffice to show that for bidder i, submitting a proxy bid of $b_i(t) = v_i$ at $t = 0$ is a (weakly) dominant strategy.

First, suppose bidder i's strategy leads to the submission of several proxy bids, $B_i(h_{t_1}) < B_i(h_{t_2}) < \cdots < B_i(h_{t_n})$, and cease to bid after $t_n \leq T$. Then the final proxy bid B_n must be in $(v_i - s, v_i]$. Otherwise, adding a proxy bid $B_i(h_{t_{n+1}}) = v_i$ at time $t_{n+1} \geq t_n$ would dominate the proposed strategy. Since $t_n \leq T$, there exists t_{n+1} between t_n and T, the ending period of the auction.[45] Hence, not bidding up to one's valuation is dominated.[46]

Due to the proxy bidding system, submitting a proxy bid $b_i(h_0) = v_i$ at time $t = 0$ would yield the same outcome if there does not exist two other bidders who have valuations $v_j, v_k \in (v_i - s, v_i)$, or if $B_i(h_{t_n}) = v_i$. However, by submitting a proxy bid equal to one's reservation price one strictly gains if there exists bidders j, k who have valuations $v_j, v_k \in (v_i - s, v_i)$, and hence, submit proxy bids $b_j = v_j$, and $b_k = v_k$ before bidder i. This is because the standing bid S_t will become either b_j or b_k, and is in $(v_i - s, v_i)$, effectively "blocking out" bidder i due to the ascending auction rule.

[45] If $t_n = T$, we simply change the last bid so that $B_i(h_{t_n}) = v_i$.

[46] This eliminates possible asymmetric equilibria such as having all but one bidder bidding zero according to some early bidding signal.

Since we assume continuous types with full support, there is a positive probability of having such opponents. Therefore, submitting a proxy bid of $b_i = v_i$ at time $t = 0$ weakly dominates the proposed strategy.

Thus, due to the ascending bidding rule, bidders race to bid early, and end up all bidding at time $t = 0$.[47]

Proof of Theorem 1. In the last auction, or stage 2, we have the standard case in which the remaining bidders bid their valuation $\beta_2 = v_i$ and the item is sold at price $v_{(3)}$ since there are $(N-1)$ bidders left.

This can be shown by solving the maximization problem of bidder (of type) i choosing her bid b_i, given other bidders' (identical) strategies b^*_{-i}. Let $b_{(2)}$ be the second-highest bid, and let the history be $h_m = h_{m,0}$, then the maximization problem is

$$\begin{aligned}&\max_{b_i} u(b_i, b_{-i})\\ &\qquad = Pr(b_i > b_{(2)}) \cdot \left(v_i - E\left[b_{(2)}\middle| b_i > b_{(2)}, h_2\right]\right)\end{aligned}$$

Suppose the (monotonic) equilibrium bid function is $\beta_2^*(v)$, we may transform the maximization problem from choosing bid b_i into choosing to "pretend to be type x" in which $\beta_2^*(x) = b_i$. Given other bidders' strategy $\beta_2^*(v_{-i})$, the maximization problem becomes:

$$\begin{aligned}\max_{x} u(x, v_{-i}) &= Pr(x > v_{(2)}) \cdot \left(v_i - E\left[\beta_2^*(V_{(2)})\middle| x > V_{(2)}, h_2\right]\right)\\ &= F_{-i}(x|v_i, h_2) \cdot v_i - \int_{\underline{v}}^{x} \beta_2^*(y_1) f_{-i}(y_1|v_i, h_2) dy_1\end{aligned}$$

where y_1 is the highest type among the bidders other than bidder i, $F_{-i}(y_1|v_i, h_2)$ is its conditional distribution function given v_i, h_2, and $f_{-i}(y_1|v_i)$ is its conditional pdf given v_i, h_2. Then, the first order condition is (at $x = v_i$)

$$\frac{\partial u}{\partial x} = f_{-i}(x|v_i, h_2) \cdot v_i - \beta_2^*(x) f_{-i}(x|v_i, h_2) = 0$$

[47] In practice, this corresponds to bidders submitting their bid when they first see the auction.

In fact,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \big(v_i - \beta_2^*(x)\big) \cdot f_{-i}(x|v_i, h_2) \\ &\geq 0 \text{ if } x < v_i \\ &\leq 0 \text{ if } x > v_i\end{aligned}$$

Therefore, $\beta_2^*(v_i) = v_i$ maximizes bidder i's expected payoff.

Knowing this, in the next-to-last stage, or the first stage, the N bidders can calculate the expected winning price of the last stage, and that would be the maximum amount bidder i is willing to pay at this stage. If the expected winning price in this stage is more than that of the next stage, they would rather wait and bid in the next stage. This can be shown by consider the utility of bidder i with valuation v_i, but bidding as if she were x in auction 1, and knowing she would bid as if she were v_i in auction 2.[48] Let $Y_1 > Y_2 > \cdots > Y_{(N-1)}$ be the order statistics of the valuations of the $(N-1)$ bidders $j \neq i$, and

$$Y_1\Big|_V \sim f_1(y_1|v_i), \qquad Y_1, Y_2\Big|_V \sim f_2(y_1, y_2|v_i)$$

then

$$\begin{aligned}u(x|v_i) = &\int_{\underline{v}}^{x} \Big[v_i - \beta_1(y_1)\Big] f_1(y_1|v_i) dy_1 \\ &+ \int_x^{\overline{v}} \int_{\underline{v}}^{\min\{v_i, y_1\}} \Big[v_i - y_2\Big] f_{12}(y_1, y_2|v_i) dy_2 dy_1\end{aligned}$$

Hence,

$$\frac{\partial u}{\partial x} = \Big[v_i - \beta_1(x)\Big] f_1(x|v_i) - \int_{\underline{v}}^{\min\{v_i, x\}} \Big[v_i - y_2\Big] f_{12}(x, y_2|v_i) dy_2 = 0$$

at $x = v_i$ if

$$\begin{aligned}\beta_1(v_i) &= \frac{1}{f_1(v_i|v_i)} \int_{\underline{v}}^{v_i} y_2 f_{12}(v_i, y_2|v_i) dy_2 \\ &= E\big[Y_2 \big| Y_1 = v_i, V_i = v_i\big] = E\big[V_{(3)} \big| V_{(1)} = v_i, V_{(2)} = v_i\big]\end{aligned}$$

[48] This is by subgame perfection.

Moreover, for $x < v_i$, $\min\{v_i, x\} = x$, and

$$\frac{\partial u}{\partial x} = v_i\Big[f_1(x|v_i) - \underbrace{\int_{\underline{v}}^{x} f_{12}(x, y_2|v_i)dy_2}_{f_1(x|v_i)}\Big] - \beta_1(x)f_1(x|v_i) + \int_{\underline{v}}^{x} y_2 f_{12}(x, y_2|v_i)dy_2$$

$$= -\beta_1(x)f_1(x|v_i) + \int_{\underline{v}}^{x} y_2 f_{12}(x, y_2|v_i)dy_2 \geq 0$$

since for $x < v_i$,

$$\begin{aligned}\beta_1(x) &= E\big[Y_2\big|Y_1 = x, V_i = x\big] \\ &\leq E\big[Y_2\big|Y_1 = x, V_i = v_i\big] \\ &= \frac{1}{f_1(x|v_i)}\int_{\underline{v}}^{x} y_2 f_{12}(x, y_2|v_i)dy_2\end{aligned}$$

On the other hand, for $x > v_i$, $\min\{v_i, x\} = v_i$, and

$$\frac{\partial u}{\partial x} = v_i\Big[\underbrace{f_1(x|v_i) - \int_{\underline{v}}^{v_i} f_{12}(x, y_2|v_i)dy_2}_{\int_{v_i}^{x} y_2 f_{12}(x,y_2|v_i)dy_2}\Big] - \beta_1(x)f_1(x|v_i) + \int_{\underline{v}}^{x} y_2 f_{12}(x, y_2|v_i)dy_2$$

$$= -\beta_1(x)f_1(x|v_i) + \int_{\underline{v}}^{x} y_2 f_{12}(x, y_2|v_i)dy_2 \geq 0$$

since for $x > v_i$,

$$\begin{aligned}\beta_1(x) &= E\big[Y_2\big|Y_1 = x, V_i = x\big] \\ &\geq E\big[Y_2\big|Y_1 = x, V_i = v_i\big] \\ &= \frac{1}{f_1(x|v_i)}\int_{\underline{v}}^{x} y_2 f_{12}(x, y_2|v_i)dy_2\end{aligned}$$

Thus, the unique symmetric perfect Bayesian Nash equilibrium for this stage is to bid

$$\beta_1(v_i) = E\Big[Y_2\Big|Y_1 = v_i, V_i = v_i\Big] = E\Big[V_{(3)}\Big|V_{(1)} = v_i, V_{(2)} = v_i\Big].$$

The item is won by the bidder with the highest valuation among the N bidders, which has valuation $v_{(1)}$ and pays the second-highest bid (submitted by the bidder with valuation $v_{(2)}$):

$$\beta_1(v_{(2)}) = E\Big[V_{(3)}\Big|V_{(1)} = v_{(2)}, V_{(2)} = v_{(2)}\Big].$$

Note that the expected revenue of stage 1 is, by affiliation,

$$\begin{aligned} E\Big\{\beta_1(V_{(2)})\Big\} &= E_{\tilde{v}_{(1)},\tilde{v}_{(2)}}\Big\{E\Big[V_{(3)}\Big|V_{(1)} = \tilde{v}_{(2)}, V_{(2)} = \tilde{v}_{(2)}\Big]\Big\} \\ &\leq E_{\tilde{v}_{(1)},\tilde{v}_{(2)}}\Big\{E\Big[V_{(3)}\Big|V_{(1)} = \tilde{v}_{(1)}, V_{(2)} = \tilde{v}_{(2)}\Big]\Big\} \\ &= E\Big[V_{(3)}\Big]. \end{aligned}$$

Hence, for $M = 2$, the revenue of the first auction is lesser or equal to that of the last auction. In particular, if valuations are independent, we have revenue equivalence, since

$$E\Big[V_{(3)}\Big|V_{(1)} = \tilde{v}_{(2)}, V_{(2)} = \tilde{v}_{(2)}\Big] = E\Big[V_{(3)}\Big|V_{(1)} = \tilde{v}_{(1)}, V_{(2)} = \tilde{v}_{(2)}\Big].$$

In fact, when $M = 2$, the information structure plays no role since the last auction is the standard case where the dominant strategy for everybody is to bid their valuation.

Proof of Lemma 1. We prove the three properties one-by-one:

1. For $v = (v_1, \cdots, v_n)$, $v' = (v'_1, \cdots, v'_n)$, and joint pdf $f(v_1, \cdots, v_n)$, we have (by affiliation)

   $$f(v \vee v')f(v \wedge v') \geq f(v)f(v').$$

   Consider $z = (y, v) = (y, v_1, \cdots, v_n)$, $v' = (y', v') = (y', v'_1, \cdots, v'_n)$, and

   $$\begin{aligned} g(z) = g(y, v_1, \cdots, v_n) &= f(v_1, \cdots, v_n) \text{ if } y = x_1 \\ &= 0 \text{ if } y \neq x_1 \end{aligned}$$

   Then we need to show that $g(z \vee z')g(z \wedge z') \geq g(z)g(z')$.

   For $x_1 = y$ and $x'_1 = y'$,

   $$\begin{aligned} g(z \vee z')g(z \wedge z') &= f(v \vee v')f(v \wedge v') \\ &\geq f(v)f(v') = g(z)g(z') \end{aligned}$$

   For $x_1 \neq y$ or $x'_1 \neq y'$, $g(z)g(z') = 0$ and the inequality is trivial.

2. Since there is no waiting cost, coming back at $t = T$ to bid (according to the then current history $h_{m,T}$) is costless. Also, since $M = 2$, we need not worry about information updating in the second and last stage. Hence, there is no loss of submitting a bid.

   However, if there were information revealed before the last minute T, bidders can gain by bidding at T, acting accordingly.

3. Since $v_1, \cdots, v_n$ are affiliated, for $s = v_j$, $v_1, \cdots, v_n, s$ are affiliated by part 1.

   Moreover, consider the order statistics of $v_1, \cdots, v_n$, namely $v_{(1)}, \cdots, v_{(n)}$. Then $v_{(1)}, \cdots, v_{(n)}, s$ are affiliated as well since their joint pdf (conditional on $s = v_j$) is

   $$n! \cdot g(v_{(1)}, \cdots, v_{(n)}, s | s = v_j) \cdot 1_{v_{(1)} \geq \cdots \geq v_{(n)}}.$$

   Hence, for $\min\{V_{(M+1)} + s, V_i\}$, a nondecreasing function of $V_1, \cdots, V_n$, we may apply Theorem 5 of Milgrom and Weber (1982)[49] to obtain that

   $$E\Big[\min\{V_{(M+1)} + s, V_i\}\Big| V_{(m)} = v_i, V_{(m+1)} = v_i, V_i = v_i, V_j \geq a\Big]$$

   is nondecreasing in a.

   Therefore we have established that any bid submitted would update the conditional expectation upwards (or unchanged), if the inferred belief is that only bidder types higher than $a > \underline{v}$ would do such action.

---

[49]Milgrom and Weber (1982) stated several important properties of affiliation. The one used here is

Theorem 5. *Let $Z_1, \cdot, Z_k$ be affiliated and let H be any nondecreasing function. Then the function h defined by*

$$h(a_1, b_1; \cdots; a_k, b_k) = E\Big[H(Z_1, \cdots, Z_k)\Big| a_1 \leq Z_1 \leq b_1, \cdots, a_k \leq Z_k \leq b_k\Big]$$

is nondecreasing in all of its arguments.

Proof of Theorem 2. Suppose all other opponents bid according to the last minute bidding strategy. Then, since nobody else bids, there is no information to update, and hence, playing the *repeated sealed-bid second price auction equilibrium,* $\beta_1(v_i) = E\Big[V_{(3)}\Big|V_{(1)} = v_i, V_{(2)} = v_i, h_{1,0}\Big]$, according to your ex ante information is your best response at the last minute $t = T$ of stage 1, because if one deviates, that would trigger other bidders to infer that that bidder has valuation $v_i \in [a, \overline{v}]$, and by part 3 of lemma 1, would raise their last minute bid to

$$\begin{aligned} &E\Big[V_{(3)}\Big|V_{(m)} = v_j, V_{(m+1)} = v_j, V_j = v_j, a < V_i < \overline{v}\Big] \\ &\geq E\Big[V_{(3)}\Big|V_{(m)} = v_j, V_{(m+1)} = v_j, V_j = v_j\underline{v} < V_i < \overline{v}\Big]. \end{aligned}$$

Proof of Theorem 3. Suppose there is another symmetric equilibrium with undominated strategies where all bidders play $\tilde{\beta}_1(v_i; h_{1,t})$ containing bidding before the last minute. By weak dominance of last minute bidding (Property 2 of Lemma 1), we know that people will revise their bids at the last minute, and hence, do update their information according to Bayes' rule, using the last minute history $\widehat{h_{1,T}}$.

For player i, consider deviating to solely using the *last minute bidding strategy:* wait and bid

$$\beta_1\Big(v_i, \overline{h_{1,T}}\Big) = E\Big[V_{(3)}\Big|V_{(1)} = v_i, V_{(2)} = v_i, \overline{h_{1,T}}\Big]$$

only at the last minute according to the then last minute history $\overline{h_{m,T}}$. The others' last minute bids are now

$$\tilde{\beta}_1(v_j; \overline{h_{1,T}}) = E\Big[V_{(3)}\Big|V_{(1)} = v_i, V_{(2)} = v_j, \overline{h_{1,T}}\Big]$$

Since the auction rule (Assumption 4c) specifies that later proxy bids must be strictly higher than previous ones, and it is weakly dominated to bid above one's valuation, after seeing bidder j submit a proxy bid, bidder i's beliefs are updated to be (at least)

$$v_j \geq b_j \geq S_t \geq 0.$$

Thus, as $h_{1,t}$ (strictly) increases, the conditional expectation

$$E\Big[V_{(3)}\Big|V_{(1)} = v_i, V_{(2)} = v_i, h_{1,t}\Big]$$

is nondecreasing.

since player i has "retracted" all of her bids that were in $h_{1,T}$ except the last minute one. Hence, we have

$$\begin{aligned}\tilde{\beta}_1(v_j;\overline{h_{1,T}}) &= E\Big[V_{(3)}\Big|V_{(1)} = v_i, V_{(2)} = v_j, \overline{h_{1,T}}\Big] \\ &\leq E\Big[V_{(3)}\Big|V_{(1)} = v_i, V_{(2)} = v_j, \widehat{h_{1,T}}\Big] = \tilde{\beta}_1(v_j;\widehat{h_{1,T}}).\end{aligned}$$

Therefore, the expected payoffs for player i goes up (or stays constant) when she deviates to the last minute bidding strategy. However, that means that $\tilde{\beta}_1(v_j;\overline{h_{1,T}})$ is a (weakly) dominated strategy, and thus, should be ruled out by weak dominance.

Proof of Lemma 2. First note that for bidder i, bidding $b_k > v_i$ is strictly dominated by bidding $b_k = v_i$ since the former adds a positive probability of winning and getting a negative payoff (for paying $b_k > v_i$).

Furthermore, subgame perfection requires that after bidder i bid b_k at time t_k, bidder $j \neq i$ with valuation v_j has to submit a *counter bid* $b_{k+1} \geq b_k + s$ at time $\big(t_k + \overline{T}\big)$, as long as $b_k + s \leq v_j$, if there were no other bids submitted during t_k and $\big(t_k + \overline{T}\big)$, or, equivalently, bidding history satisfies $h_{t_k} = h_{t_k+\overline{T}}$. Hence, as the standing bid increases, lesser bidders would be required to submit a counter bid since the condition to counter bid, $v_j \geq b_n + s$, is more likely to fail. Regardless of the bidding path, as the standing bid increases, counter bids would cease to exist if and only if the only bidder that might satisfy the condition $v_j \geq b_n + s$ is the the one who submitted the last bid b_n itself. In that case, no one will submit a counter bid during the going-going-gone period $\Big(t_n, t_n + \overline{T}\Big]$, and thus, the auction ends.

Since there is a minimum increment s, and the bidding sequence $\{b_k\}$ is strictly increasing, there exist a finite $\overline{n}$ such that $b_{\overline{n}} > v_{(2)} - s$, submitted by either the highest bidder (with valuation $v_{(1)}$) or the second-highest bidder (with valuation $v_{(2)}$).

If it was the highest bidder submitting $b_{\overline{n}}$, the second-highest bidder would not counter bid since $b_{\overline{n}} + s > v_{(2)}$.

However, as long as $s < v_{(1)} - v_{(2)}$,[50] the highest bidder can always counter any possible bid b_k of the second-highest bidder, which is at most $v_{(2)}$. Hence, the bidding sequence is finite, and will end when someone bids b_n in $\Big((v_{(2)} - s), (v_{(2)} + s)\Big]$.

This yields an efficient outcome, and as $s \to 0$, the winning price b_n converges to $v_{(2)}$, the winning price in the (irrevocable) button auction.

Proof of Proposition 2. Facing opponents' bidding strategy: Bid $s, 2s, 3s, \cdots, ks, \cdots$ at the time $\tilde{T}, 2\tilde{T}, \cdots, k\tilde{T}, \cdots$ is as if facing a raising price clock with the "tick" defined by the minimal increment s. Hence, the question is whether one can "jump bid" after a certain history to stop this price clock earlier than following the equilibrium bidding strategy, which is to bid according to this price clock, since by jump bidding to b, one can "block" out opponents who have valuations lower than $b + s$.

We now consider the case where at a current price ks, a bidder who has $(k+2)s \leq v_i < (k+3)s$ jump bidding to $(k+2)s$ at time $(k+1)\tilde{T}$, instead of bidding $(k+1)s$ as "scheduled." (Note that if the current price is below ks, the "loss" is even larger for jump bidding, while the "gains" of squeezing out opponents remain the same. Thus, the case considered here provides the upper bound of gains from deviation.) The gains are from blocking out $y_1 \in [v, (k+3)s)$ since $(k+2)s + s = (k+3)s$. The losses occur when $y_1 \in [ks, (k+1)s]$, and hence, one could have won and paid $(k+1)s$ instead. Therefore, in equilibrium, we require

$$
\begin{aligned}
\Delta u = Pr\Big(v < y_1 < (k+3)s\Big) \cdot \big[v - (k+2)s\big] \\
+ Pr\Big(ks < y_1 < (k+1)s\Big) \cdot \big[v - (k+2)s - v + (k+1)s\big] \\
= \Big[F_2\Big((k+3)s\Big) - F_2(v)\Big] \cdot [v - (k+2)s] + \Big[F_2\Big((k+1)s\Big) - F_2(ks)\Big] \cdot (-s) \\
\leq 0
\end{aligned}
$$

[50]If $s \geq v_{(1)} - v_{(2)}$, we still get approximate efficiency since the bidder with $v_{(2)}$ wins the item and further gains from trade is $v_{(1)} - v_{(2)}$, which is at most s.

Or,

$$
\begin{aligned}
\frac{\frac{F_2((k+3)s)-F_2(v)}{(k+3)s-v}}{\frac{F_2((k+1)s)-F_2(ks)}{s}} &\leq \frac{s}{(k+3)s-v} \cdot \frac{s}{v-(k+2)s} \\
&= \frac{1}{\left[(k+3)-\frac{v}{s}\right] \cdot \left[\frac{v}{s}-(k+2)\right]} \\
&= \frac{1}{\frac{1}{4}-\left[\frac{v}{s}-(k+\frac{5}{2})\right]^2} \\
&\leq 4 \text{ for } \frac{v}{s} \in [k+2, k+3].
\end{aligned}
$$

Since $F_2' = f_2$ exists for all $v_{(2)}$, by the mean value theorem, there exists $\xi_1 \in [v, (k+3)s]$ and $\xi_2 \in [ks, (k+1)s]$ such that

$$
\frac{f_2(\xi_1)}{f_2(\xi_2)} \;=\; \frac{\frac{F_2((k+3)s)-F_2(v)}{(k+3)s-v}}{\frac{F_2((k+1)s)-F_2(ks)}{s}} \;\leq\; 4
$$

because by assumption,

$$
\frac{f_2(y+\eta)}{f_2(y)} \leq 4 \text{ for all } y \in [\underline{v}, \overline{v}] \text{ and } \eta \in [s, 3s].
$$

Thus, the requirement of non-deviation is satisfied, and the proposed symmetric strategy consists an equilibrium.

As $s \to 0$, the smoothness assumption requires smaller and smaller ranges of η, and as $\eta \to 0$,

$$
\frac{f_2(y+\eta)}{f_2(y)} \to 1 < 4
$$

trivially.

Finally, when $s \to 0$, the outcome is bidders jointly pushing up the standing price by s every $\tilde{T}$. Letting $\tilde{T} = o(s)$ (so that the clock stops in finite time), we have constructed a "price clock" which is equivalent to what we see in a button auction, and the cease-to-bid criteria of $v_i \geq ks = b_i(ks)$ is equivalent to the dropping out strategy.

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