Exercise 18.14: $\quad \text{Max } f(x,y,z) = xyz$

(Example 18.9) $\quad$ s.t. $p_x x + p_y y + p_z z \le I$

$x \ge 0,\ y \ge 0,\ z \ge 0$

Use K-T.

$$\tilde{\mathcal{L}} = xyz - \lambda\left[p_x x + p_y y + p_z z - I\right]$$

FOC:
$$\boxed{A}\begin{cases} \dfrac{\partial \tilde{\mathcal{L}}}{\partial x} = yz - \lambda p_x \le 0, & x \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial x} = 0 \\ \dfrac{\partial \tilde{\mathcal{L}}}{\partial y} = xz - \lambda p_y \le 0, & y \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial y} = 0 \\ \dfrac{\partial \tilde{\mathcal{L}}}{\partial z} = xy - \lambda p_z \le 0, & z \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial z} = 0 \end{cases}$$

$\boxed{B}$ $\quad I - p_x x - p_y y - p_z z \ge 0,$

$\lambda \cdot (I - p_x x - p_y y - p_z z) = 0$

NDCQ: $\nabla g = (p_x, p_y, p_z) \neq \vec{0}$

① If $\lambda = 0$: $\begin{cases} yz \le 0 \Rightarrow yz = 0 \\ xz \le 0 \Rightarrow xz = 0 \\ xy \le 0 \Rightarrow xy = 0 \end{cases}$ $\Rightarrow$ $\underline{xyz = 0}$

(2 of $x, y, z = 0$)

② If $\lambda > 0$: $p_x x + p_y y + p_z z = I$

$\Rightarrow x, y, z$ at least one $\neq 0$

If $x > 0$: $yz = \lambda p_x > 0 \Rightarrow y, z > 0$

$\Rightarrow$ "=" for all $\boxed{A}$

Hence, $\lambda = \dfrac{yz}{p_x} = \dfrac{xz}{p_y} = \dfrac{xy}{p_z}$

$\Rightarrow y = \dfrac{p_x}{p_y} \cdot x,\ z = \dfrac{p_x}{p_z} \cdot x$

$\therefore I = p_x \cdot x + p_x x + p_x \cdot x = 3 p_x x$

$\Rightarrow x = \dfrac{I}{3p_x},\ y = \dfrac{I}{3p_y},\ z = \dfrac{I}{3p_z}$ $\quad \underline{xyz > 0}$ #

Example: Given Production cost $C(y) \in C^1$ satisfies $C' > 0$

Firm choose output $y \in \mathbb{R}_+$, advertising cost $a \in \mathbb{R}_+$

to maximize Revenue $R(y,a) \in C^1$, $\dfrac{\partial R}{\partial a} > 0$, $R(0,a) = 0$

without letting profit drop below $m > 0$

$$\max_{y,a} R(y,a)$$
s.t. $\pi = R(y,a) - C(y) - a \ge m$
$y \ge 0,\ a \ge 0$

$$\tilde{\mathcal{L}} = R(y,a) - \lambda\left[m - R(y,a) + C(y) + a\right]$$ If NDCQ holds,

F.O.C.:

$\dfrac{\partial \tilde{\mathcal{L}}}{\partial y} = \dfrac{\partial R}{\partial y}(1+\lambda) - \lambda C'(y) \le 0, \quad y \cdot \left(\dfrac{\partial \tilde{\mathcal{L}}}{\partial y}\right) = 0$

$\dfrac{\partial \tilde{\mathcal{L}}}{\partial a} = \dfrac{\partial R}{\partial a}(1+\lambda) - \lambda \le 0, \quad a \cdot \left(\dfrac{\partial \tilde{\mathcal{L}}}{\partial a}\right) = 0$

$\dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda} = R(y,a) - C(y) - a - m \ge 0, \quad \lambda \cdot \left[\dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda}\right] = 0$

Example: $\frac{\partial R}{\partial a} > 0$, $R(0,a)=0$, $C'>0$

F.O.C.:

$$\boxed{A}\begin{cases} \frac{\partial \tilde{\mathcal{L}}}{\partial y} = \frac{\partial R}{\partial y}(1+\lambda) - \lambda C'(y) \overset{\le}{=} 0, & y\cdot\left(\frac{\partial \tilde{\mathcal{L}}}{\partial y}\right) = 0 \\ \frac{\partial \tilde{\mathcal{L}}}{\partial a} = \frac{\partial R}{\partial a}(1+\lambda) - \lambda \overset{\le}{=} 0, & a\cdot\left(\frac{\partial \tilde{\mathcal{L}}}{\partial a}\right) = 0 \end{cases}$$

$$\boxed{B}\ \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda} = R(y,a) - C(y) - a - m \overset{\ge}{=} 0, \quad \lambda\cdot\left[\frac{\partial \tilde{\mathcal{L}}}{\partial \lambda}\right] = 0$$

① suppose $\lambda = 0 \Rightarrow \frac{\partial \tilde{\mathcal{L}}}{\partial a} = \frac{\partial R}{\partial a} > 0$ (contradicting $\boxed{A}$-a) $\Rightarrow \lambda > 0$.

② $\lambda > 0 \Rightarrow \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda} = 0$, or $R(y,a) - C(y) - a = m$

③ If $y > 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial R}{\partial y}(1+\lambda) - \lambda C'(y) = 0 \Rightarrow \frac{\partial R}{\partial y} = \frac{\lambda}{1+\lambda} C'(y)$

Hence, $\frac{\partial R}{\partial y} - C'(y) = \frac{-1}{1+\lambda} C'(y) < 0$

i.e. y^* is larger than $y^{\max \Pi}$ (that satisfies $\frac{\partial R}{\partial y} - C'(y) = 0$)

Wait, does NDCQ hold?

$$g(y,a) = m - \Pi = m - R(y,a) + C(y) + a \le 0$$

$$Dg = \nabla g = \left(\frac{\partial g}{\partial y}, \frac{\partial g}{\partial a}\right) = \left(-\frac{\partial R}{\partial y} + C'(y), -\frac{\partial R}{\partial a} + 1\right)$$

$$= \vec{0} \text{ if } \underbrace{\frac{\partial R}{\partial y}(y^{**}, a^{**})}_{\text{MR}} = \underbrace{C'(y^{**})}_{\text{MC}} \ \& \ \frac{\partial R}{\partial a}(y^{**}, a^{**}) = 1$$

i.e. NDCQ fails are MR = MC & MR of advertising = 1.

Hence, if $\exists\,(y^{**}, a^{**})$ such that MR = MC & MR of advertising = 1, this point is also a candicate for the solution since it fails NDCQ (and hence, need not satisfy F.O.C.)

Meaning of Lagrange Multiplier

Firm offers activity service $x_1, \cdots, x_n$ to generate profit $f(x_1, \cdots, x_n)$
to hold these activities, the firm needs to use input $j = 1, \cdots, k$
These inputs are subject to resource constraints

$$\left.\begin{array}{l} g_1(x_1, \cdots, x_n) = \text{amount of input } 1 \\ \vdots \\ g_k(x_1, \cdots, x_n) = \text{amount of input } k \end{array}\right\} \text{required to hold } x_1, \cdots, x_n$$

$$\Rightarrow \left.\begin{array}{l} g_1(x_1, \cdots, x_n) \le a_1 \\ \vdots \\ g_k(x_1, \cdots, x_n) \le a_k \end{array}\right\} \text{is the resource constraint for} \left\{\begin{array}{l} \text{input } 1 \\ \vdots \\ \text{input } k \end{array}\right.$$

$$\Rightarrow \frac{\partial}{\partial a_j}\left[f(x_1^*(a), \cdots, x_n^*(a)) \right] = \lambda_j^*(a) = \text{shadow price of input } j$$

$= $ change in optimal profit by adding one more unit of input j.