Constrained Optimization: F.O.C.

經濟學是蝦密? Optimal allocation of scarce resources

如: $x_1 > 0,\ x_2 > 0$

$$\text{Max } \underbrace{f(x_1, \cdots, x_n)}_{\text{Objective function}} \quad \text{s.t.} \begin{cases} g_1(x_1, \cdots, x_n) \le b_1 \\ \quad \vdots \\ g_k(x_1, \cdots, x_n) \le b_k \end{cases} \quad \text{(inequality constraints)}$$

$$\begin{cases} h_1(x_1, \cdots, x_n) = c_1 \\ \quad \vdots \\ h_m(x_1, \cdots, x_n) = c_m \end{cases} \quad \text{(equality constraints)}$$

EX: 1. 廠商問題: (Profit Maximization of a competitive firm)

Total Revenue $= p \cdot y$

$$\text{Max } \pi(x_1, \cdots, x_n) = p \cdot f(x_1, \cdots, x_n) - \underbrace{(w_1 x_1 + w_2 x_2 + \cdots + w_n x_n)}_{\text{Total Cost}}$$

s.t. $pf(x_1, \cdots, x_n) - (w_1 x_1 + w_2 x_2 + \cdots + w_n x_n) \ge 0$ (nonnegative profits)

$g_1(x_1, \cdots, x_n) \le b_1, \ \cdots, \ g_k(x_1, \cdots, x_n) \le b_k$ (availability of inputs)

$x_1 \ge 0,\ x_2 \ge 0, \cdots, x_n \ge 0$

2. 消費者問題: (Utility Maximization)

$$\max\ U(x_1, \cdots, x_n)$$

s.t. $p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \le I$ (budget constraint)

$x_1 \geq 0,\ x_2 \geq 0,\ \cdots,\ x_n \geq 0$ (nonnegative constraints)

3. 加時間限制式的消費者問題: (Utility Maximization with Labor/Leisure)

$$\max\ U(x_1, \cdots, x_n, l_1)$$

s.t. $p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \le I = I' + w l_0,\ \ l_0 + l_1 = 24$

$x_1 \geq 0,\ x_2 \geq 0,\ \cdots,\ x_n \geq 0,\ l_0 \geq 0,\ l_1 \geq 0$

(time constraint)

Homework: Exercise 18.4, 18.9, 18.13, 18.14, 18.15, 18.20, 18.21

Equality Constraint(s): $h(x_1, x_2) = c$

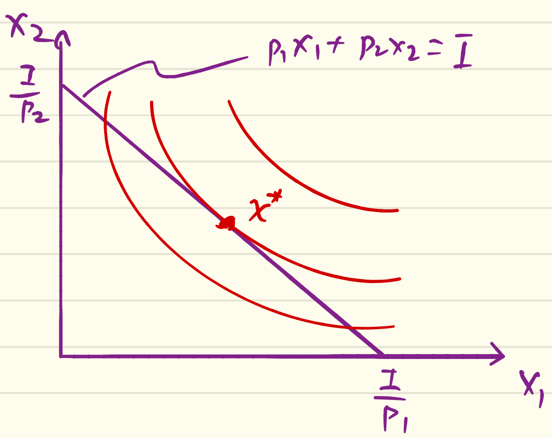
消費者問題: $\boxed{\max\limits_{x_1, x_2} f(x_1, x_2) \quad \text{s.t.}\ \ p_1 x_1 + p_2 x_2 = I}$

兩變數、一條限制等式.

($x_1 \geq 0,\ x_2 \geq 0$ 先忽略)

$h(x_1, x_2) = p_1 x_1 + p_2 x_2 = I = c$

圖形解法:



找 indifference curve (無差異曲線) 和 budget constraint (預算限制式) 的切點

$$\text{相切} \Rightarrow \frac{-\frac{\partial f}{\partial x_1}(x^*)}{\frac{\partial f}{\partial x_2}(x^*)} = \frac{-\frac{\partial h}{\partial x_1}(x^*)}{\frac{\partial h}{\partial x_2}(x^*)}$$

$$\frac{-\frac{\partial f}{\partial x_1}(x^*)}{\boxed{\frac{\partial f}{\partial x_2}(x^*)}} = \frac{\boxed{-\frac{\partial h}{\partial x_1}(x^*)}}{\frac{\partial h}{\partial x_2}(x^*)} \Rightarrow \frac{\frac{\partial f}{\partial x_1}(x^*)}{\frac{\partial h}{\partial x_1}(x^*)} = \frac{\frac{\partial f}{\partial x_2}(x^*)}{\frac{\partial h}{\partial x_2}(x^*)} = \mu$$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x_1}(x^*) - \mu \frac{\partial h}{\partial x_1}(x^*) = 0 \quad \text{—①} \\ \frac{\partial f}{\partial x_2}(x^*) - \mu \frac{\partial h}{\partial x_2}(x^*) = 0 \quad \text{—②} \end{cases}$$

加上 $h(x_1, x_2) = c$ —③ 共三條方程式

可解 x_1, x_2, μ

其實 ①, ②, ③ 就是 Lagrangian Function 的 F.O.C., μ 是 Lagrange Multiplier.

$$\mathcal{L}(x_1, x_2, \mu) = f(x_1, x_2) - \mu[h(x_1, x_2) - c]$$

FOC: $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f}{\partial x_1}(x^*) - \mu \frac{\partial h}{\partial x_1}(x^*) = 0$ —① $\frac{\partial \mathcal{L}}{\partial \mu} = h(x_1, x_2) - c = 0$ —③

$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial f}{\partial x_2}(x^*) - \mu \frac{\partial h}{\partial x_2}(x^*) = 0$ —②

也就是說: 兩變數、一條限制等式的 Constrained optimization

(變成) "=" 三變數的 Unconstrained optimization

但書: 需要 $\frac{\partial h}{\partial x_1}$, $\frac{\partial h}{\partial x_2}$ 不同時為 0!! ⇒ Constraint Qualification (CQ)

EX: 前面幾個例子 $h(x_1, x_2)$ 都是線性函數, 故 CQ 自動成立.

故, 我們有下面這個定理:

Thm 18.1 Let f, h be C^1 functions of two variables.

Suppose $x^* = (x_1^*, x_2^*)$ is a solution of the problem

$$\max_{x_1, x_2} \{ f(x_1, x_2) \mid h(x_1, x_2) = c \}.$$

If x^* is not a critical point of h, then $\exists\, \mu^* \in \mathbb{R}$ s.t.

$\frac{\partial \mathcal{L}}{\partial x_1} = 0$, $\frac{\partial \mathcal{L}}{\partial x_2} = 0$, $\frac{\partial \mathcal{L}}{\partial \mu} = 0$ at (x_1^*, x_2^*, μ^*) for

$$\mathcal{L}(x_1, x_2, \mu) \equiv f(x_1, x_2) - \mu[h(x_1, x_2) - c].$$

更一般地來說，考慮 gradient (梯度) $\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix}$, $\nabla h(x) = \begin{pmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \end{pmatrix}$,

也就是和 Level set 垂直的向量，則 "f, h 在 x^* 相切" 也就代表兩 gradient 平行：

$$\underline{\nabla f(x^*) = \mu^* \nabla h(x^*)} \equiv ① \,\&\, ② \quad , \text{ multiplier } \mu^* \in \mathbb{R}$$

<u>Remark</u>：極大 or 極小要看 SOC (在 19.3)，Thm 的證明在 19.6.

EX: 消費者問題：
$$\boxed{\begin{aligned} &\max_{x_1, x_2} \ f(x_1, x_2) = x_1 x_2 \\ &\text{s.t.} \quad p_1 x_1 + p_2 x_2 = I \end{aligned}} \equiv x_1 + 4x_2 = 16$$

(Sol) $\nabla h(x) = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \neq 0$ at all x. $\Rightarrow$ CQ satisfied.

$$\mathcal{L}(x_1, x_2, \mu) = x_1 x_2 - \mu [x_1 + 4x_2 - 16]$$

FOC:
$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = x_2 - \mu = 0 & \Rightarrow \mu = x_2 = \frac{1}{4} x_1 \\ \frac{\partial \mathcal{L}}{\partial x_2} = x_1 - 4\mu = 0 & \Downarrow \qquad (\text{唯一可能}) \\ \frac{\partial \mathcal{L}}{\partial \mu} = -x_1 - 4x_2 + 16 = 0 & \Rightarrow x_1 + 4\left(\frac{1}{4}x_1\right) - 16 = 2x_1 - 16 = 0 \Rightarrow \begin{cases} x_1 = 8 \\ x_2 = 2 = \mu \end{cases} \end{cases}$$

EX: ~~消費者問題~~: $\max\limits_{x_1, x_2} f(x_1, x_2) = \underline{x_1^2 x_2}$

$$\text{s.t. } (x_1, x_2) \in C^h = \{(x_1, x_2) \mid \underline{2x_1^2 + x_2^2 = 3}\}$$

(Sol) $\frac{\partial h}{\partial x_1} = 4x_1$, $\frac{\partial h}{\partial x_2} = 2x_2$ $\Rightarrow$ CQ satisfied for all $(x_1, x_2) \neq (0, 0)$

$$\mathcal{L}(x_1, x_2, \mu) = x_1^2 x_2 - \mu\left[2x_1^2 + x_2^2 - 3\right]$$

FOC:
$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 x_2 - 4\mu x_1 = 0 = 2x_1(x_2 - 2\mu) & \Rightarrow x_1 = 0 \text{ or } x_2 = 2\mu \\ \frac{\partial \mathcal{L}}{\partial x_2} = x_1^2 - 2\mu x_2 = 0 & \Rightarrow x_1^2 = 2\mu x_2 \\ \frac{\partial \mathcal{L}}{\partial \mu} = -2x_1^2 - x_2^2 + 3 = 0 & \Rightarrow 2x_1^2 + x_2^2 = 3 \end{cases}$$

(i) $x_1 \neq 0$: $x_2 = 2\mu$, $x_1^2 = (2\mu)^2 = x_2^2$ $\Rightarrow$ $3x_1^2 = 3$ or $x_1 = \pm 1$, $x_2 = \pm 1$

$\Rightarrow (1, 1, 0.5), (-1, 1, 0.5), (1, -1, -0.5), (-1, -1, -0.5)$

(ii) $x_1 = 0$: $\Rightarrow x_2^2 = 3$ $\Rightarrow x_2 = \pm\sqrt{3}$, $\mu = \frac{x_1^2}{2x_2} = 0$ $\Rightarrow (0, \pm\sqrt{3}, 0)$

$f(\underline{1, 1}) = f(\underline{-1, 1}) = 1 > f(0, \pm\sqrt{3}) = 0 > f(1, -1) = f(-1, -1) = -1 \Rightarrow \underline{(\pm 1, 1)}$

<u>小結</u>: 如何解 Constrained Optimization (兩變數.一個限制等式)

1. 先檢查 CQ: $\frac{\partial h}{\partial x_1} = 0$ & $\frac{\partial h}{\partial x_2} = 0$ 為 critical point.

   如果有, 則列入可能的候選人.

2. CQ ok, 則列式 $\mathcal{L}$

   用 FOC 找出所有可能是 optimal 的 critical point

3. 比較所有候選人那一個讓 $f(x^*)$ 最大(最小).

Exercise: 用 Thm 18.2 證明算幾不等式:

(1) Maximize $x^2y^2z^2$ subject to $x^2+y^2+z^2=c^2$, c is constant.

What is this maximum value?

(2) Show that $x^2y^2z^2 \le \left[\frac{1}{3}(x^2+y^2+z^2)\right]^3$, or $\sqrt[3]{x^2y^2z^2} \le \frac{x^2+y^2+z^2}{3}$.

(3) Extend your results to any $n \in \mathbb{N}$:

$$(x_1^2 \cdot x_2^2 \cdot \cdots \cdot x_n^2)^{\frac{1}{n}} \le \frac{1}{n}\left[x_1^2+x_2^2+\cdots+x_n^2\right]$$

with equality iff $x_1^2 = x_2^2 = \cdots = x_n^2$.

Several Equality Constraints:

$$\text{Max} \quad f(x_1,\cdots,x_n)$$

$$\text{s.t.} \quad C_h = \{x=(x_1,\cdots,x_n) \mid h_1(x)=a_1,\ h_2(x)=a_2,\ \cdots,\ h_m(x)=a_m\}$$

How can we extend CQ in the simple case? $\left(\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}\right) \neq (0,0)$

Consider Jacobian $Dh(\hat{x}) = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(\hat{x}) & \cdots & \frac{\partial h_1}{\partial x_n}(\hat{x}) \\ \frac{\partial h_2}{\partial x_1}(\hat{x}) & \cdots & \frac{\partial h_2}{\partial x_n}(\hat{x}) \\ \vdots & & \vdots \\ \frac{\partial h_m}{\partial x_1}(\hat{x}) & \cdots & \frac{\partial h_m}{\partial x_n}(\hat{x}) \end{pmatrix} = \begin{pmatrix} Dh_1 \\ Dh_2 \\ \vdots \\ Dh_m \end{pmatrix}$

$\Rightarrow x$ is a critical point if $\text{rank}(Dh(\hat{x})) < m$.

$h = (h_1, \cdots, h_m)$ satisfies NDCQ (Non-Degenerate Constraint Qualification) at $\hat{x}$ if $\text{rank}(Dh(\hat{x})) = m$.

i.e. Vectors $\underset{\underset{Dh_1}{\parallel}}{\frac{\partial h_1}{\partial x}}, \underset{\underset{Dh_2}{\parallel}}{\frac{\partial h_2}{\partial x}}, \cdots, \underset{\underset{Dh_m = \left(\frac{\partial h_m}{\partial x_1}, \cdots, \frac{\partial h_m}{\partial x_n}\right)}{\parallel}}{\frac{\partial h_m}{\partial x}}$ are linearly independent

s.t. $H = \{v = a_1(Dh_1) + a_2(Dh_2) + \cdots + a_m(Dh_m) : a_i \in \mathbb{R}\}$ has dimension $= m$.

Linear dependence vs. Linear independence :

Vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m (\in \mathbb{R}^n)$ are linearly dependent if

$\exists\, a_1, \cdots, a_m \in \mathbb{R}$ such that $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m = \vec{0}$

If no such a_i exists, $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent.

$\Rightarrow$ If $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent, the space it spans

$H = \{v = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m : a_1, \cdots, a_m \in \mathbb{R}\}$ has dimension $= m$.

$h=(h_1,\cdots,h_m)$ satisfies NDCQ (Non-Degenerate Constraint Qualification) at $\hat{x}$ if $\operatorname{rank}(Dh(\hat{x}))=m$ (or $Dh_1,\cdots,Dh_m$ are linearly independent).

Then, we have

<u>Thm 18.2</u> Let $f, h_1,\cdots,h_m \in C^1$, consider

$$\text{Max } f(x_1,\cdots,x_n)$$

$$\text{s.t. } C_h=\{x=(x_1,\cdots,x_n) \mid h_1(x)=a_1,\ h_2(x)=a_2,\cdots,\ h_m(x)=a_m\}$$

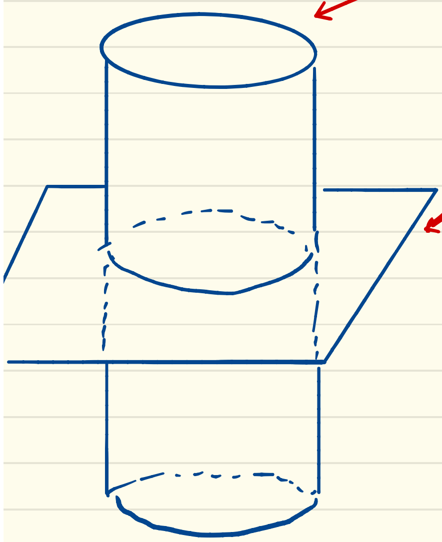
Suppose $x^*=(x_1^*,\cdots,x_n^*)\in C_h$ and satisfies NDCQ.

If x^* is a (local) max or min of f on C_h, then there exists $\mu^*=(\mu_1^*,\cdots,\mu_m^*)$ such that (x^*,μ^*) is a critical point of

$$\mathcal{L}(x,\mu)=f(x)-\mu_1[h_1(x)-a_1]-\mu_2[h_2(x)-a_2]-\cdots-\mu_m[h_m(x)-a_m]$$

i.e. $\frac{\partial\mathcal{L}}{\partial x_1}(x^*,\mu^*)=0,\ \cdots\ \frac{\partial\mathcal{L}}{\partial x_n}(x^*,\mu^*)=0;\quad \frac{\partial\mathcal{L}}{\partial \mu_1}(x^*,\mu^*)=0,\ \cdots,\ \frac{\partial\mathcal{L}}{\partial h_m}(x^*,\mu^*)=0$

Example (18.6): Max xyz



s.t. $h_1(x,y,z)=x^2+y^2=1$

$h_2(x,y,z)=x+z=1$

$$\Rightarrow Dh(x,y,z)=\begin{pmatrix}2x & 2y & 0\\ 1 & 0 & 1\end{pmatrix}$$

$\Rightarrow$ rank $=2$ unless $x=y=0$ (But this violates h_1)

$\therefore$ NDCQ satisfied for all $(x,y,z)\in C_h$

$$\mathcal{L} = xyz - \mu_1(x^2+y^2-1) - \mu_2(x+z-1)$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial x} = yz - 2\mu_1 x - \mu_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = xz - 2\mu_1 y = 0 \Rightarrow \mu_1 = \frac{xz}{2y}$$

$$\frac{\partial \mathcal{L}}{\partial z} = xy - \mu_2 = 0 \Rightarrow \mu_2 = xy$$

$$\Rightarrow yz - \frac{xz}{y} = xy \Rightarrow (y^2-x^2)z = xy^2$$

$$\frac{\partial \mathcal{L}}{\partial \mu_1} = x^2+y^2-1 = 0 \Rightarrow y^2 = 1-x^2$$

$$\frac{\partial \mathcal{L}}{\partial \mu_2} = x+z-1 = 0 \Rightarrow z = 1-x$$

$$(1-2x^2)(\cancel{1-x}) = x\overset{1+x}{(\cancel{1-x^2})}$$

$$\Rightarrow 1-x=0 \quad \text{or} \quad 1-2x^2 = x+x^2 \Rightarrow 3x^2+x-1=0$$

$$\Rightarrow x^* = \frac{1}{6}(-1\pm\sqrt{13}) \approx \begin{cases} +0.4343 \\ -0.7676 \end{cases}, \quad y^* = 1-x^2 = \begin{cases} \pm 0.9008 \\ \pm 0.6409 \end{cases}, \quad z^* = 1-x^* = \begin{cases} 0.5657 \\ 1.7676 \end{cases}$$

Check to see xyz maximized at $x^* = -0.7676$, $y^* = -0.6409$, $z^* = 1.7676$

Inequality Constraint(s): $\overset{g}{\cancel{X}}(x_1, x_2) \overset{\le b}{\cancel{=c}}$

消費者問題:

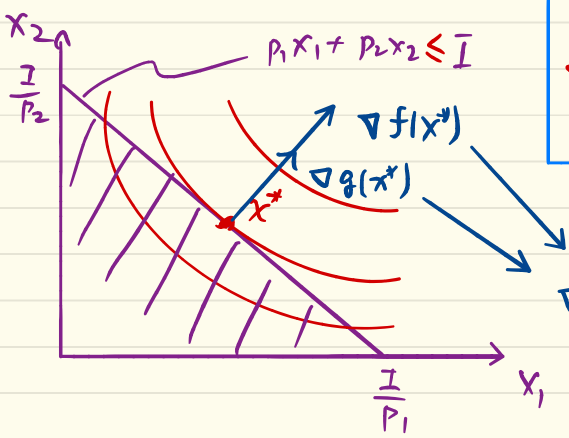
$$\boxed{\max_{x_1,x_2} f(x_1,x_2) \quad \text{s.t.} \quad p_1x_1 + p_2x_2 \le I}$$

兩變數、一條限制不等式

($x_1 \ge 0$, $x_2 \ge 0$ 先忽略)

$$g(x_1,x_2) = p_1x_1 + p_2x_2 \le I = b$$

圖形解法:



$$\boxed{\begin{cases} g \\ b \\ \lambda \end{cases} \text{for inequality}, \quad \begin{cases} h \\ c \\ \mu \end{cases} \text{for equality}}$$

$\nabla f(x^*) = \lambda \nabla g(x^*)$ if binding ("=")

而且 $\lambda \ge 0$，因為 $\nabla f(x^*)$ 指向 f 增加最大的方向，如果指向 $g(x^*) \le b$ (即 $\lambda < 0$!) 則可往該方向移動、增加 $f(x_1,x_2)$ 但又保持 $g(x_1,x_2) \le b$，跟 x^* 是 Max 矛盾!

因此設 $\mathcal{L}(x_1, x_2, \mu) = f(x_1, x_2) - \lambda[g(x_1, x_2) - b]$ (而且仍要符合 CQ)

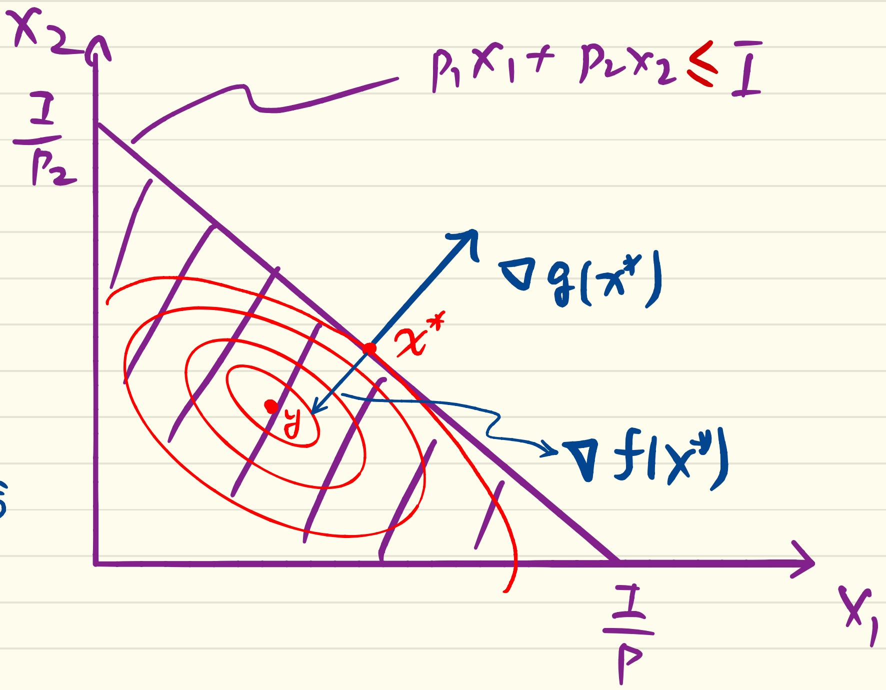
FOC: $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f}{\partial x_1}(x^*) - \lambda \frac{\partial g}{\partial x_1}(x^*) = 0$, $\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial f}{\partial x_2}(x^*) - \lambda \frac{\partial g}{\partial x_2}(x^*) = 0$

但是等一下：如果 constraint not binding 怎麼辦??

圖形解法：

若 y 才是 Max，則 constraint 是 not binding ("<")，表示 y 也是 local (unconstraint) max.

(的確，$\nabla f(x^*)$ 指向 $g(x_1, x_2) \le b$，故可往該方向增加 f，但又符合 $g(x_1, x_2) \le b$.)



若 y 才是 Max，則 constraint 是 not binding ("<")，表示 y 也是 local max.

因此設 $\mathcal{L}(x_1, x_2, \mu) = f(x_1, x_2) - \lambda[g(x_1, x_2) - b]$ 亦可，只要 $\lambda = 0$!!

因 FOC:
$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f}{\partial x_1}(x^*) - \lambda \frac{\partial g}{\partial x_1}(x^*) = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial f}{\partial x_2}(x^*) - \lambda \frac{\partial g}{\partial x_2}(x^*) = 0 \end{cases}$$
即為
$$\begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{cases}$$

也就是說，我們需要 $\lambda \cdot [g(x_1, x_2) - b] = 0$

(即 $\lambda = 0$ or $g(x_1, x_2) - b = 0$)

因此設 $\mathcal{L}(x_1, x_2, \mu) = f(x_1, x_2) - \lambda[g(x_1, x_2) - b]$，則：

FOC: $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f}{\partial x_1}(x^*) - \lambda \frac{\partial g}{\partial x_1}(x^*) = 0$ —①

$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial f}{\partial x_2}(x^*) - \lambda \frac{\partial g}{\partial x_2}(x^*) = 0$ —②

$\lambda \cdot [g(x_1, x_2) - b] = 0$ —③

故,我們有下面這個定理:

Thm 18.3 Let f, g be C^1 functions of two variables.

Suppose $x^* = (x_1^*, x_2^*)$ is a solution of the problem

$$\max_{x_1, x_2} \{ f(x_1, x_2) \mid g(x_1, x_2) \leq b \}.$$

If $g(x^*) = b$, then x^* is not a critical point of g.

(i.e. $\frac{\partial g}{\partial x_1}(x^*) \neq 0$ or $\frac{\partial g}{\partial x_2}(x^*) \neq 0$)

Then for $\mathcal{L}(x_1, x_2, \mu) \equiv f(x_1, x_2) - \lambda [g(x_1, x_2) - b]$

$\exists\ \lambda^* \in \mathbb{R}$ s.t. (a) $\frac{\partial \mathcal{L}}{\partial x_1} = 0$, (b) $\frac{\partial \mathcal{L}}{\partial x_2} = 0$ at (x_1^*, x_2^*, μ^*)

and (c) $\lambda^* \cdot [g(x^*) - b] = 0$

(d) $\lambda^* \geq 0$

(e) $\frac{\partial \mathcal{L}}{\partial \lambda} = g(x^*) - b \leq 0$

Remark: 比較 Thm 18.1 ("=") 和 Thm 18.3 ("≤"):

1. 兩者都要求 $\frac{\partial \mathcal{L}}{\partial x_1} = 0$, $\frac{\partial \mathcal{L}}{\partial x_2} = 0$
2. $\frac{\partial \mathcal{L}}{\partial \mu} = h(x^*) - c = 0 \longrightarrow \frac{\partial \mathcal{L}}{\partial \lambda} = g(x^*) - b \leq 0$ & $\lambda [g(x^*) - b] = 0$
3. CQ $\longrightarrow$ 當 constraint bind 時才要求 CQ.
4. $\mu^* \in \mathbb{R} \longrightarrow \lambda^* \geq 0$
5. min 做法同 max $\longrightarrow \lambda^* \geq 0$ for max, $\lambda^* \leq 0$ for min
   (見 18.5)

但其實 $\lambda \cdot [g(x^*) - b] = 0$ 可被看作

If $\lambda > 0$, then $g(x^*) = b$ (binding). 而簡化為 equality 版本

Example: $\max\ U(x_1, x_2)$ (Assume $U(x_1, x_2) \in C^1$

s.t. $p_1 x_1 + p_2 x_2 \leq I$ $\frac{\partial U}{\partial x_1} > 0$, $\frac{\partial U}{\partial x_2} > 0$)

$\mathcal{L} = U(x_1, x_2) - \lambda [p_1 x_1 + p_2 x_2 - I]$, CQ ok.

FOC: $\frac{\partial U}{\partial x_1} - \lambda p_1 = 0$ $\Rightarrow \lambda = \frac{\frac{\partial U}{\partial x_1}}{p_1} > 0$ $\therefore p_1 x_1 + p_2 x_2 - I = 0$

$\frac{\partial U}{\partial x_2} - \lambda p_2 = 0$ (equality case !!

$\lambda \cdot [p_1 x_1 + p_2 x_2 - I] = 0$