VOLUME INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

A point in space may be located in terms of many different coordinate systems. Cartesian coordinates were used above. In the following, we consider the use of two other coordinate systems—cylindrical coordinates and spherical coordinates.

cylindrical coordinates

polar coordinates

Cylindrical Coordinates Cartesian coordinates locate a point in space by three distances x, y, and z (refer to Fig. 8.5). *Cylindrical coordinates* locate a point in space by two distances r, z and an angle θ (see Fig. 8.6). Cylindrical coordinates consist of *polar coordinates* (r, θ) in the xy plane and a third coordinate (z) measured perpendicular to the xy plane. Thus, in cylindrical coordinates, a point P in space is located by the coordinates (r, θ, z), whereas, in rectangular coordinates, it is located by the coordinates (x, y, z), as shown in Fig. 8.6.

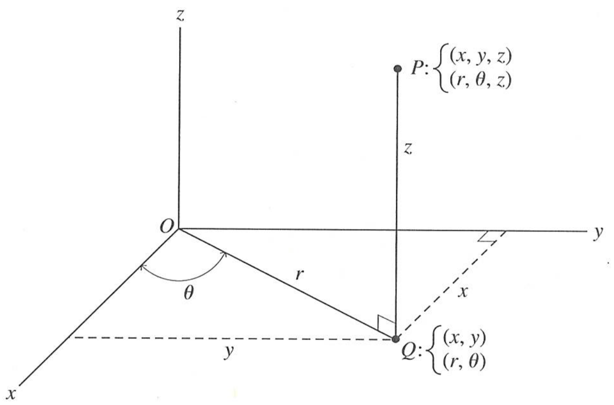


Figure 8.6 Cylindrical coordinates (r, θ, z).

The values of the two Cartesian coordinates x and y and the two polar coordinates r and θ are related by the equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r^2 = x^2 + y^2$$

In cylindrical coordinates, cylinders whose axes lie along the z axis and planes that either contain the z axis or are perpendicular to the z axis have especially simple equations (see Fig. 8.7). When a body or volume involves these shapes, cylindrical coordinates may be the best coordinates to use to evaluate Eqs. (8.5) and (8.8). Note that in cylindrical coordinates the equation $r = a$ describes not just a circle in the xy plane but an entire cylinder perpendicular to the xy plane (Fig. 8.7a). Note also that in cylindrical coordinates the z axis is given by the equation $r = 0$. The equation $\theta = \theta_0$ describes the plane that contains the z axis and that forms an angle θ_0 with the positive x axis (Fig. 8.7b). The equation $z = z_0$ describes the plane that is perpendicular to the z axis and that passes through the point $(0, 0, z_0)$ (Fig. 8.7c).

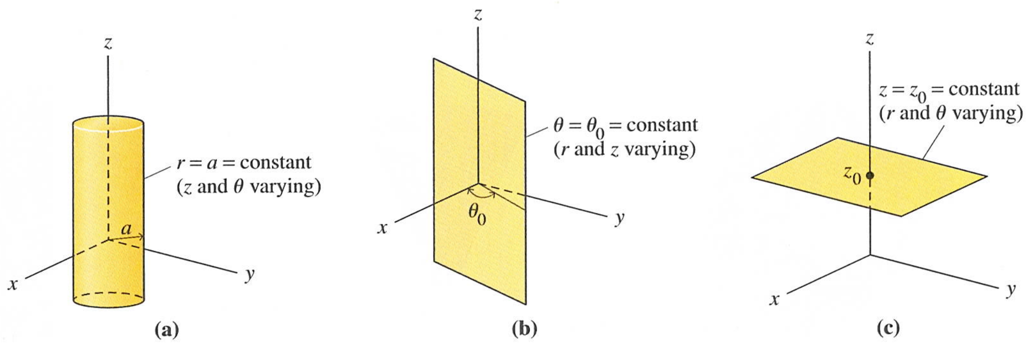


Figure 8.7 Special surfaces: **(a)** cylinder, $r = a$; **(b)** plane containing z axis, $\theta = \theta_0$; **(c)** plane perpendicular to z axis, $z = z_0$.

spherical coordinates

Spherical Coordinates To locate a point P in space, we can also use *spherical coordinates* consisting of two angles θ and ϕ and a distance ρ (see Fig. 8.8). The coordinate ρ is the distance from the origin O to point P. The coordinate ϕ is the angle that the line OP forms with the z axis; it lies in the range $0 \le \phi \le \pi$. The coordinate θ is the same as in cylindrical coordinates (refer to Fig. 8.6).[3]

In spherical coordinates, the equation $\rho = a$ describes a sphere of radius a with center at the origin O (Fig. 8.9a). The equation $\phi = \phi_0$ describes a cone whose vertex lies at the origin O and whose axis lies along the z axis. For $\phi_0 < \pi/2$, the cone opens upward (extends from the origin along the positive z axis). For $\pi/2 < \phi_0 < \pi$, the cone opens downward (extends from the origin along the negative z axis). The special cases $\phi_0 = 0$, $\phi_0 = \pi/2$, and $\phi_0 = \pi$ represent the positive z axis, the xy plane, and the negative z axis, respectively. As in cylindrical coordinates, the equation $\theta = \theta_0$ describes the plane that contains the z axis and that forms an angle θ_0 with the positive x axis (Fig. 8.9c). When a problem involves these shapes, spherical coordinates may be the best coordinates to use to evaluate Eqs. (8.5) and (8.8).

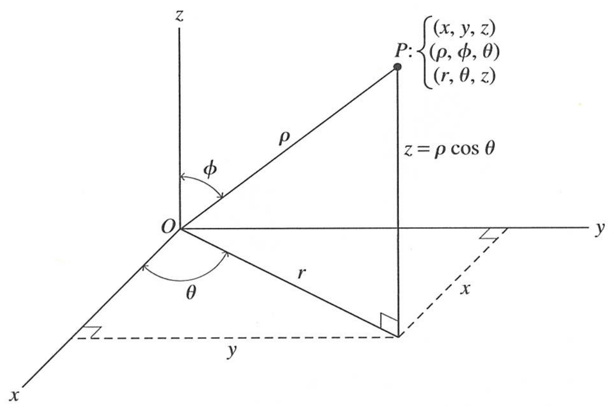


Figure 8.8 Spherical coordinates (ρ, θ, ϕ).

3. In some books, the angles θ and ϕ are reversed. Be alert to this fact when using spherical coordinates.

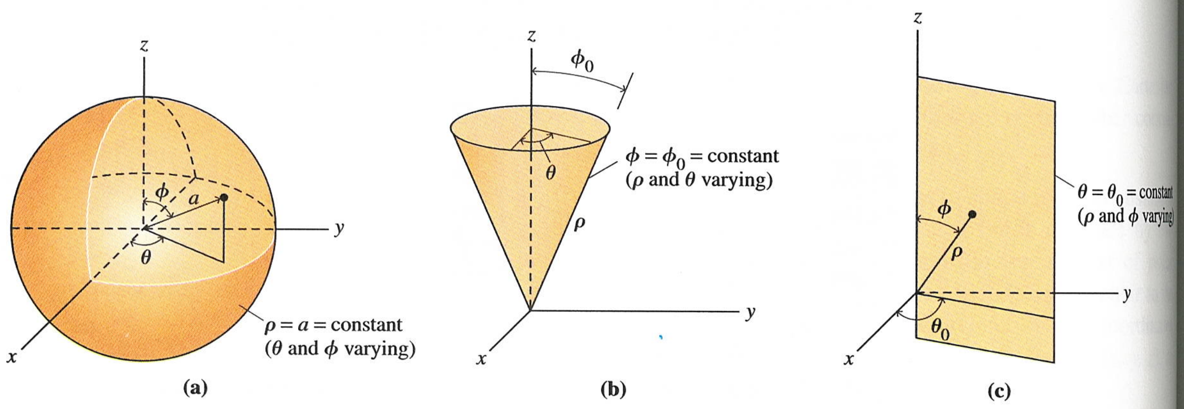


Figure 8.9 Special surfaces: **(a)** sphere with center at origin, $\rho = a$; **(b)** cone with vertex at origin, $\phi = \phi_0$; **(c)** plane that contains the z axis, $\theta = \theta_0$.

Equations relating Cartesian, cylindrical, and spherical coordinates are (refer to Figs. 8.6 and 8.8)

$$r = \rho \sin \phi$$

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

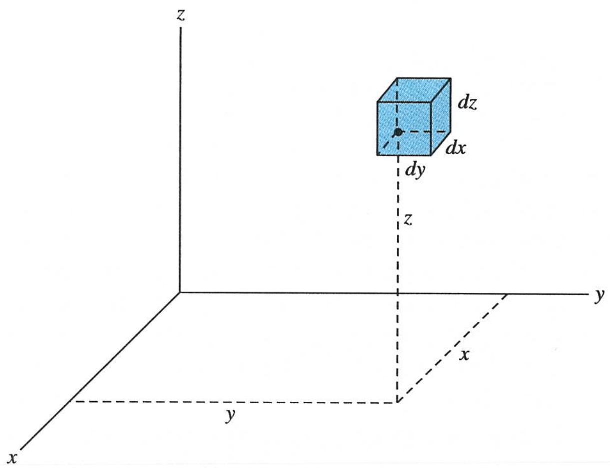
For the xy plane, $z = 0$, $\phi = \pi/2$, and $r = \rho$.

Volume Elements and Integrals In Cartesian coordinates, the volume element dV in terms of dx, dy, and dz is (see Fig. 8.10)

$$dV = dx\,dy\,dz \quad \text{(b)}$$

and the volume is

$$V = \iiint dx\,dy\,dz \quad \text{(c)}$$



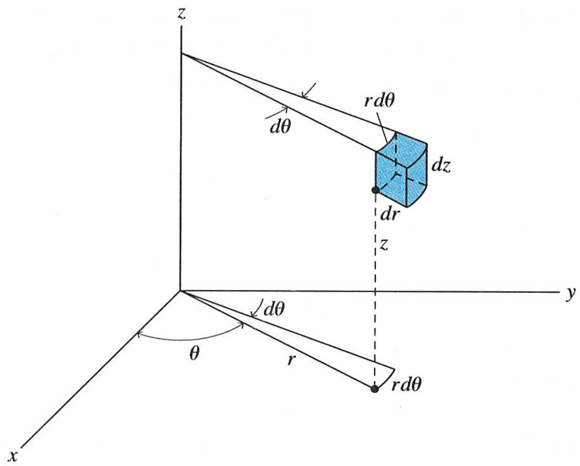


Figure 8.11 Volume element in cylindrical coordinates.

In cylindrical coordinates, in terms of dr, $d\theta$, and dz, the volume element dV is (see Fig. 8.11)

$$dV = (dr)(r\,d\theta)(dz) = r\,dr\,d\theta\,dz \tag{d}$$

and the volume is

$$V = \iiint r\,dr\,d\theta\,dz \tag{e}$$

In spherical coordinates, in terms of $d\rho$, $d\theta$, $d\phi$, the volume element dV is (see Fig. 8.12)

$$dV = (d\rho)(\rho \sin\phi\,d\theta)(\rho\,d\phi) = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi \tag{8.9}$$

and the volume is

$$V = \iiint \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

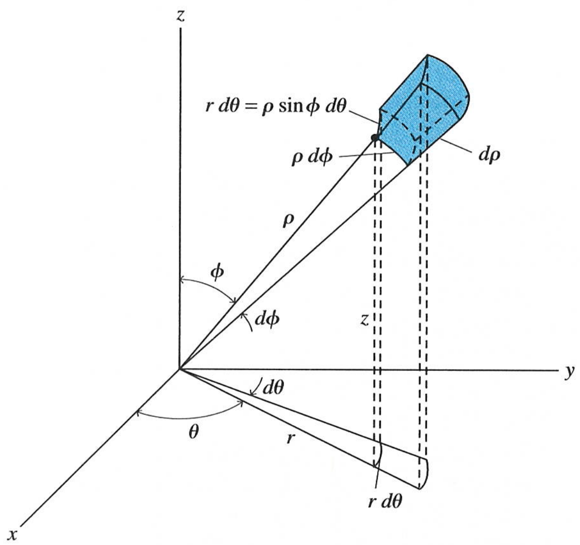


Figure 8.12 Volume element in spherical coordinates.

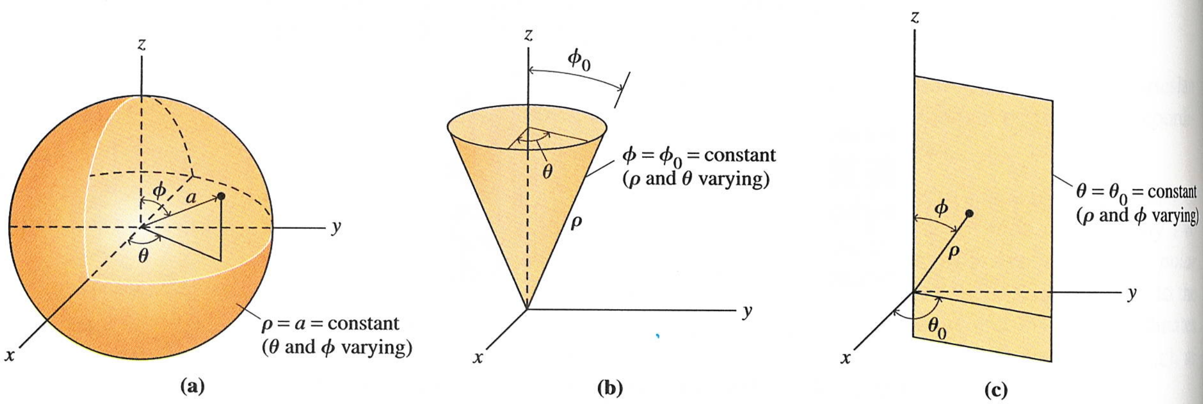


Figure 8.9 Special surfaces: **(a)** sphere with center at origin, $\rho = a$; **(b)** cone with vertex at origin, $\phi = \phi_0$; **(c)** plane that contains the z axis, $\theta = \theta_0$.

Equations relating Cartesian, cylindrical, and spherical coordinates are (refer to Figs. 8.6 and 8.8)

$$r = \rho \sin \phi$$

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

For the xy plane, $z = 0$, $\phi = \pi/2$, and $r = \rho$.

Volume Elements and Integrals In Cartesian coordinates, the volume element dV in terms of dx, dy, and dz is (see Fig. 8.10)

$$dV = dx\,dy\,dz \tag{b}$$

and the volume is

$$V = \iiint dx\,dy\,dz \tag{c}$$

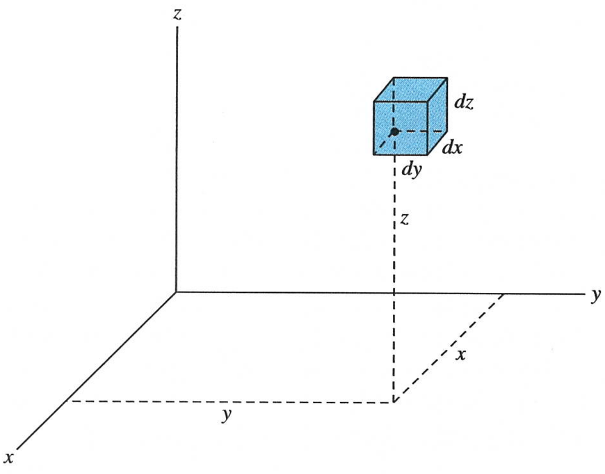


Figure 8.10 Volume element in Cartesian coordinates.

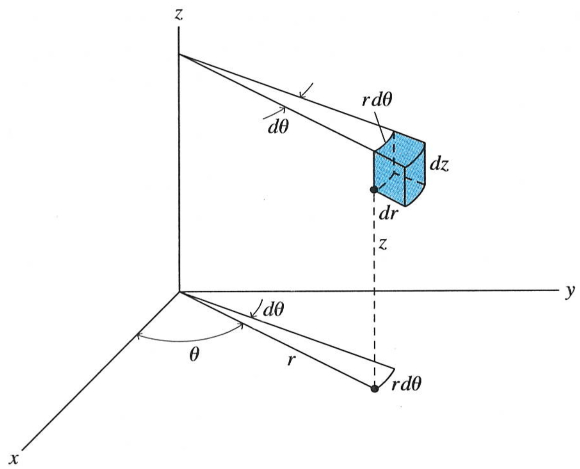


Figure 8.11 Volume element in cylindrical coordinates.

In cylindrical coordinates, in terms of dr, $d\theta$, and dz, the volume element dV is (see Fig. 8.11)

$$dV = (dr)(r\,d\theta)(dz) = r\,dr\,d\theta\,dz \tag{d}$$

and the volume is

$$V = \iiint r\,dr\,d\theta\,dz \tag{e}$$

In spherical coordinates, in terms of $d\rho$, $d\theta$, $d\phi$, the volume element dV is (see Fig. 8.12)

$$dV = (d\rho)(\rho\sin\phi\,d\theta)(\rho\,d\phi) = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi \tag{8.9}$$

and the volume is

$$V = \iiint \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

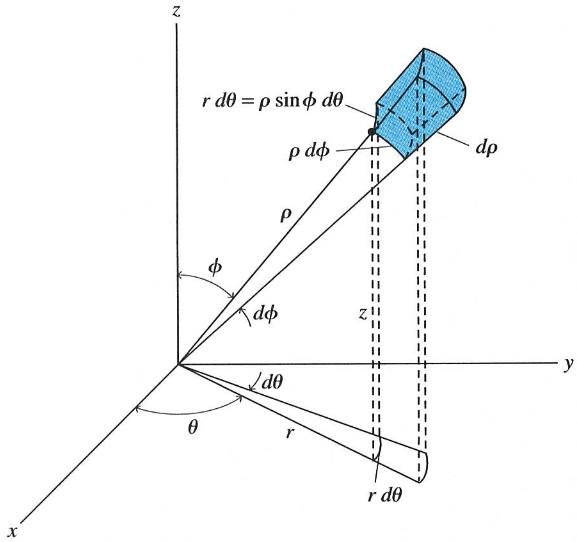


Figure 8.12 Volume element in spherical coordinates.

The integration principles discussed in this section may be used to determine the coordinates of the center of gravity or the centroid of a body. The application of these principles is summarized in the following Problem-Solving Technique and examples.

PROBLEM-SOLVING TECHNIQUE

Center of Gravity and Centroid of a Body

To determine the center of gravity or the centroid of a body by integration:

1. Sketch the body approximately to scale.
2. Choose an appropriate coordinate system for the analysis, depending on the shape of the body, and show it in the sketch (from Step 1). If the body has flat plane boundaries, Cartesian coordinates are efficient. If the body has curved boundaries, cylindrical or spherical coordinates may be better.
3. Utilize symmetry properties of the body to help determine the coordinates of the center of gravity or centroid. If the body has a line or a plane of symmetry, the center of gravity (or centroid) lies on that line or plane. If the body has two lines or planes of symmetry, the center of gravity (or centroid) lies at the intersection of the lines or on the intersection line of the planes. A coordinate axis or plane should be chosen to coincide with the line or plane of symmetry (see Step 2).
4. Choose a differential element of weight dW of the body (or a differential element of volume dV), and express the coordinates of dW (or of dV) in terms of the chosen coordinate system.
5. Determine the weight W (or the volume V) of the body by integration of dW (or dV). For the weight integration, use the appropriate specific weight γ (constant or variable).
6. Determine the first moments of weight (or of volume) by integration of Eqs. (8.6) [or of Eqs. (8.8)].
7. Divide the first moments of weight (or volume) (Step 6) by the weight W (or volume V) to determine the coordinates of the center of gravity (or of the centroid).
8. Locate the center of gravity (or centroid) in the sketch from Step 1. Check to make certain that the location makes sense. In addition, it is a good idea to check your results by solving the problem a different way, if you have time. For example, use a different differential element or a different coordinate system.

Example 8.3

Centroid of a Cone

Problem Statement Consider a solid cone of arbitrary cross section, with its vertex at the origin and its base perpendicular to the x axis (see Fig. E8.3). Determine the distance $\bar{x}$ of the cone's centroid from the yz plane.

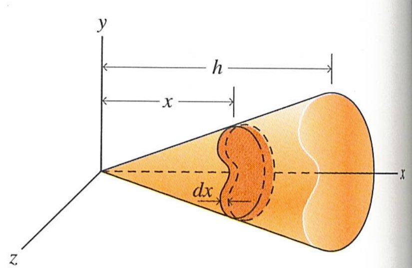


Figure E8.3

Solution Since all the cross sections of the cone have the same shape as the base and since the linear dimensions (width, diameter, etc.) of any cross section are proportional to x, the area S of a