2. If $A \in \mathsf{W}$ and $B \in \mathsf{W}$, then $A^t = A$ and $B^t = B$. Thus $(A+B)^t = A^t + B^t = A + B$, so that $A + B \in \mathsf{W}$.

3. If $A \in \mathsf{W}$, then $A^t = A$. So for any $a \in F$, we have $(aA)^t = aA^t = aA$. Thus $aA \in \mathsf{W}$.

The examples that follow provide further illustrations of the concept of a subspace. The first three are particularly important.

Example 1

Let n be a nonnegative integer, and let $\mathsf{P}_n(F)$ consist of all polynomials in $\mathsf{P}(F)$ having degree less than or equal to n. Since the zero polynomial has degree -1, it is in $\mathsf{P}_n(F)$. Moreover, the sum of two polynomials with degrees less than or equal to n is another polynomial of degree less than or equal to n, and the product of a scalar and a polynomial of degree less than or equal to n is a polynomial of degree less than or equal to n. So $\mathsf{P}_n(F)$ is closed under addition and scalar multiplication. It therefore follows from Theorem 1.3 that $\mathsf{P}_n(F)$ is a subspace of $\mathsf{P}(F)$. ♦

Example 2

Let $\mathsf{C}(R)$ denote the set of all continuous real-valued functions defined on R. Clearly $\mathsf{C}(R)$ is a subset of the vector space $\mathcal{F}(R, R)$ defined in Example 3 of Section 1.2. We claim that $\mathsf{C}(R)$ is a subspace of $\mathcal{F}(R, R)$. First note that the zero of $\mathcal{F}(R, R)$ is the constant function defined by $f(t) = 0$ for all $t \in R$. Since constant functions are continuous, we have $f \in \mathsf{C}(R)$. Moreover, the sum of two continuous functions is continuous, and the product of a real number and a continuous function is continuous. So $\mathsf{C}(R)$ is closed under addition and scalar multiplication and hence is a subspace of $\mathcal{F}(R, R)$ by Theorem 1.3. ♦

Example 3

An $n \times n$ matrix M is called a **diagonal matrix** if $M_{ij} = 0$ whenever $i \neq j$, that is, if all its nondiagonal entries are zero. Clearly the zero matrix is a diagonal matrix because all of its entries are 0. Moreover, if A and B are diagonal $n \times n$ matrices, then whenever $i \neq j$,

$$(A+B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0 \quad \text{and} \quad (cA)_{ij} = cA_{ij} = c0 = 0$$

for any scalar c. Hence $A + B$ and cA are diagonal matrices for any scalar c. Therefore the set of diagonal matrices is a subspace of $\mathsf{M}_{n\times n}(F)$ by Theorem 1.3. ♦

Example 4

The **trace** of an $n \times n$ matrix M, denoted $\operatorname{tr}(M)$, is the sum of the diagonal entries of M; that is,

$$\operatorname{tr}(M) = M_{11} + M_{22} + \cdots + M_{nn}.$$

It follows from Exercise 6 that the set of $n \times n$ matrices having trace equal to zero is a subspace of $\mathsf{M}_{n\times n}(F)$. ♦

Example 5

The set of matrices in $\mathsf{M}_{m\times n}(R)$ having nonnegative entries is not a subspace of $\mathsf{M}_{m\times n}(R)$ because it is not closed under scalar multiplication (by negative scalars). ♦

The next theorem shows how to form a new subspace from other subspaces.

Theorem 1.4. *Any intersection of subspaces of a vector space* V *is a subspace of* V.

Proof. Let $\mathcal{C}$ be a collection of subspaces of V, and let W denote the intersection of the subspaces in $\mathcal{C}$. Since every subspace contains the zero vector, $0 \in \mathsf{W}$. Let $a \in F$ and $x, y \in \mathsf{W}$. Then x and y are contained in each subspace in $\mathcal{C}$. Because each subspace in $\mathcal{C}$ is closed under addition and scalar multiplication, it follows that $x + y$ and ax are contained in each subspace in $\mathcal{C}$. Hence $x + y$ and ax are also contained in W, so that W is a subspace of V by Theorem 1.3. ∎

Having shown that the intersection of subspaces of a vector space V is a subspace of V, it is natural to consider whether or not the union of subspaces of V is a subspace of V. It is easily seen that the union of subspaces must contain the zero vector and be closed under scalar multiplication, but in general the union of subspaces of V need not be closed under addition. In fact, it can be readily shown that the union of two subspaces of V is a subspace of V if and only if one of the subspaces contains the other. (See Exercise 19.) There is, however, a natural way to combine two subspaces W_1 and W_2 to obtain a subspace that contains both W_1 and W_2. As we already have suggested, the key to finding such a subspace is to assure that it must be closed under addition. This idea is explored in Exercise 23.

EXERCISES

1. Label the following statements as true or false.
   (a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.
   (b) The empty set is a subspace of every vector space.
   (c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $\mathsf{W} \neq \mathsf{V}$.
   (d) The intersection of any two subsets of V is a subspace of V.

(e) An $n \times n$ diagonal matrix can never have more than n nonzero entries.
(f) The trace of a square matrix is the product of its diagonal entries.
(g) Let W be the xy-plane in R^3; that is, $\mathsf{W} = \{(a_1, a_2, 0) \colon a_1, a_2 \in R\}$. Then $\mathsf{W} = \mathsf{R}^2$.

2. Determine the transpose of each of the matrices that follow. In addition, if the matrix is square, compute its trace.

(a) $\begin{pmatrix} -4 & 2 \\ 5 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 8 & -6 \\ 3 & 4 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} -3 & 9 \\ 0 & -2 \\ 6 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 10 & 0 & -8 \\ 2 & -4 & 3 \\ -5 & 7 & 6 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & -1 & 3 & 5 \end{pmatrix}$ (f) $\begin{pmatrix} -2 & 5 & 1 & 4 \\ 7 & 0 & 1 & -6 \end{pmatrix}$

(g) $\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ (h) $\begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{pmatrix}$

3. Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in \mathsf{M}_{m\times n}(F)$ and any $a, b \in F$.

4. Prove that $(A^t)^t = A$ for each $A \in \mathsf{M}_{m\times n}(F)$.

5. Prove that $A + A^t$ is symmetric for any square matrix A.

6. Prove that $\operatorname{tr}(aA + bB) = a\operatorname{tr}(A) + b\operatorname{tr}(B)$ for any $A, B \in \mathsf{M}_{n\times n}(F)$.

7. Prove that diagonal matrices are symmetric matrices.

8. Determine whether the following sets are subspaces of R^3 under the operations of addition and scalar multiplication defined on R^3. Justify your answers.

(a) $\mathsf{W}_1 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 \colon a_1 = 3a_2 \text{ and } a_3 = -a_2\}$
(b) $\mathsf{W}_2 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 \colon a_1 = a_3 + 2\}$
(c) $\mathsf{W}_3 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 \colon 2a_1 - 7a_2 + a_3 = 0\}$
(d) $\mathsf{W}_4 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 \colon a_1 - 4a_2 - a_3 = 0\}$
(e) $\mathsf{W}_5 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 \colon a_1 + 2a_2 - 3a_3 = 1\}$
(f) $\mathsf{W}_6 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 \colon 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

9. Let W_1, W_3, and W_4 be as in Exercise 8. Describe $\mathsf{W}_1 \cap \mathsf{W}_3$, $\mathsf{W}_1 \cap \mathsf{W}_4$, and $\mathsf{W}_3 \cap \mathsf{W}_4$, and observe that each is a subspace of R^3.

10. Prove that $\mathsf{W}_1 = \{(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n \colon a_1 + a_2 + \cdots + a_n = 0\}$ is a subspace of F^n, but $\mathsf{W}_2 = \{(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n \colon a_1 + a_2 + \cdots + a_n = 1\}$ is not.

11. Is the set $\mathsf{W} = \{f(x) \in \mathsf{P}(F) \colon f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$ a subspace of $\mathsf{P}(F)$ if $n \geq 1$? Justify your answer.

12. An $m \times n$ matrix A is called **upper triangular** if all entries lying below the diagonal entries are zero, that is, if $A_{ij} = 0$ whenever $i > j$. Prove that the upper triangular matrices form a subspace of $\mathsf{M}_{m\times n}(F)$.

13. Let S be a nonempty set and F a field. Prove that for any $s_0 \in S$, $\{f \in \mathcal{F}(S, F) \colon f(s_0) = 0\}$, is a subspace of $\mathcal{F}(S, F)$.

14. Let S be a nonempty set and F a field. Let $\mathcal{C}(S, F)$ denote the set of all functions $f \in \mathcal{F}(S, F)$ such that $f(s) = 0$ for all but a finite number of elements of S. Prove that $\mathcal{C}(S, F)$ is a subspace of $\mathcal{F}(S, F)$.

15. Is the set of all differentiable real-valued functions defined on R a subspace of $\mathsf{C}(R)$? Justify your answer.

16. Let $\mathsf{C}^n(R)$ denote the set of all real-valued functions defined on the real line that have a continuous nth derivative. Prove that $\mathsf{C}^n(R)$ is a subspace of $\mathcal{F}(R, R)$.

17. Prove that a subset W of a vector space V is a subspace of V if and only if $\mathsf{W} \neq \varnothing$, and, whenever $a \in F$ and $x, y \in \mathsf{W}$, then $ax \in \mathsf{W}$ and $x + y \in \mathsf{W}$.

18. Prove that a subset W of a vector space V is a subspace of V if and only if $0 \in \mathsf{W}$ and $ax + y \in \mathsf{W}$ whenever $a \in F$ and $x, y \in W$.

19. Let W_1 and W_2 be subspaces of a vector space V. Prove that $\mathsf{W}_1 \cup \mathsf{W}_2$ is a subspace of V if and only if $\mathsf{W}_1 \subseteq \mathsf{W}_2$ or $\mathsf{W}_2 \subseteq \mathsf{W}_1$.

20.† Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in \mathsf{W}$ for any scalars $a_1, a_2, \ldots, a_n$.

21. Show that the set of convergent sequences $\{a_n\}$ (i.e., those for which $\lim_{n\to\infty} a_n$ exists) is a subspace of the vector space V in Exercise 20 of Section 1.2.

22. Let F_1 and F_2 be fields. A function $g \in \mathcal{F}(F_1, F_2)$ is called an **even function** if $g(-t) = g(t)$ for each $t \in F_1$ and is called an **odd function** if $g(-t) = -g(t)$ for each $t \in F_1$. Prove that the set of all even functions in $\mathcal{F}(F_1, F_2)$ and the set of all odd functions in $\mathcal{F}(F_1, F_2)$ are subspaces of $\mathcal{F}(F_1, F_2)$.

†A dagger means that this exercise is essential for a later section.