

Proof. Exercise. ■

Corollary 2. The vector y described in (VS 4) is unique.

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The vector 0 in (VS 3) is called the **zero vector** of V , and the vector y in (VS 4) (that is, the unique vector such that $x + y = 0$) is called the **additive inverse** of x and is denoted by $-x$.

The next result contains some of the elementary properties of scalar multiplication.

Theorem 1.2. In any vector space V , the following statements are true:

- (a) $0x = 0$ for each $x \in V$.
- (b) $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in V$.
- (c) $a0 = 0$ for each $a \in F$.

Proof. (a) By (VS 8), (VS 3), and (VS 1), it follows that

$$0x + 0x = (0 + 0)x = 0x = 0x + 0 = 0 + 0x.$$

Hence $0x = 0$ by Theorem 1.1.

(b) The vector $-(ax)$ is the unique element of V such that $ax + [-(ax)] = 0$. Thus if $ax + (-a)x = 0$, Corollary 2 to Theorem 1.1 implies that $(-a)x = -(ax)$. But by (VS 8),

$$ax + (-a)x = [a + (-a)]x = 0x = 0$$

by (a). Consequently $(-a)x = -(ax)$. In particular, $(-1)x = -x$. So, by (VS 6),

$$a(-x) = a[(-1)x] = [a(-1)]x = (-a)x.$$

The proof of (c) is similar to the proof of (a). ■

EXERCISES

1. Label the following statements as true or false.
 - (a) Every vector space contains a zero vector.
 - (b) A vector space may have more than one zero vector.
 - (c) In any vector space, $ax = bx$ implies that $a = b$.
 - (d) In any vector space, $ax = ay$ implies that $x = y$.
 - (e) A vector in F^n may be regarded as a matrix in $M_{n \times 1}(F)$.
 - (f) An $m \times n$ matrix has m columns and n rows.
 - (g) In $P(F)$, only polynomials of the same degree may be added.
 - (h) If f and g are polynomials of degree n , then $f + g$ is a polynomial of degree n .
 - (i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n .

- (j) A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero.
- (k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if they have the same value at each element of S .

2. Write the zero vector of $M_{3 \times 4}(F)$.

3. If

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

what are M_{13} , M_{21} , and M_{22} ?

4. Perform the indicated operations.

(a) $\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 5 \\ -5 & 3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix} + \begin{pmatrix} 7 & -5 \\ 0 & -3 \\ 2 & 0 \end{pmatrix}$

(c) $4 \begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix}$

(d) $-5 \begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix}$

(e) $(2x^4 - 7x^3 + 4x + 3) + (8x^3 + 2x^2 - 6x + 7)$

(f) $(-3x^3 + 7x^2 + 8x - 6) + (2x^3 - 8x + 10)$

(g) $5(2x^7 - 6x^4 + 8x^2 - 3x)$

(h) $3(x^5 - 2x^3 + 4x + 2)$

Exercises 5 and 6 show why the definitions of matrix addition and scalar multiplication (as defined in Example 2) are the appropriate ones.

5. Richard Gard ("Effects of Beaver on Trout in Sagehen Creek, California," *J. Wildlife Management*, 25, 221-242) reports the following number of trout having crossed beaver dams in Sagehen Creek.

Upstream Crossings

	Fall	Spring	Summer
Brook trout	8	3	1
Rainbow trout	3	0	0

Downstream Crossings

	Fall	Spring	Summer
Brook trout	9	1	4
Rainbow trout	3	0	0
Brown trout	1	1	0

Record the upstream and downstream crossings in two 3×3 matrices, and verify that the sum of these matrices gives the total number of crossings (both upstream and downstream) categorized by trout species and season.

6. At the end of May, a furniture store had the following inventory.

	Early American	Spanish	Mediterranean	Danish
Living room suites	4	2	1	3
Bedroom suites	5	1	1	4
Dining room suites	3	1	2	6

Record these data as a 3×4 matrix M . To prepare for its June sale, the store decided to double its inventory on each of the items listed in the preceding table. Assuming that none of the present stock is sold until the additional furniture arrives, verify that the inventory on hand after the order is filled is described by the matrix $2M$. If the inventory at the end of June is described by the matrix

$$A = \begin{pmatrix} 5 & 3 & 1 & 2 \\ 6 & 2 & 1 & 5 \\ 1 & 0 & 3 & 3 \end{pmatrix},$$

interpret $2M - A$. How many suites were sold during the June sale?

7. Let $S = \{0, 1\}$ and $F = R$. In $\mathcal{F}(S, R)$, show that $f = g$ and $f + g = h$, where $f(t) = 2t + 1$, $g(t) = 1 + 4t - 2t^2$, and $h(t) = 5t + 1$.
8. In any vector space V , show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in F$.
9. Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2(c).
10. Let V denote the set of all differentiable real-valued functions defined on the real line. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3.

11. Let $V = \{0\}$ consist of a single vector 0 and define $0 + 0 = 0$ and $c0 = 0$ for each scalar c in F . Prove that V is a vector space over F . (V is called the **zero vector space**.)
12. A real-valued function f defined on the real line is called an **even function** if $f(-t) = f(t)$ for each real number t . Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.
13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over R with these operations? Justify your answer.

14. Let $V = \{(a_1, a_2, \dots, a_n) : a_i \in C \text{ for } i = 1, 2, \dots, n\}$; so V is a vector space over C by Example 1. Is V a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?
15. Let $V = \{(a_1, a_2, \dots, a_n) : a_i \in R \text{ for } i = 1, 2, \dots, n\}$; so V is a vector space over R by Example 1. Is V a vector space over the field of complex numbers with the operations of coordinatewise addition and multiplication?
16. Let V denote the set of all $m \times n$ matrices with real entries; so V is a vector space over R by Example 2. Let F be the field of rational numbers. Is V a vector space over F with the usual definitions of matrix addition and scalar multiplication?
17. Let $V = \{(a_1, a_2) : a_1, a_2 \in F\}$, where F is a field. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in V$, define

$$c(a_1, a_2) = (a_1, 0).$$

Is V a vector space over F with these operations? Justify your answer.

18. Let $V = \{(a_1, a_2) : a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over R with these operations? Justify your answer.

19. Let $V = \{(a_1, a_2) : a_1, a_2 \in R\}$. Define addition of elements of V coordinatewise, and for (a_1, a_2) in V and $c \in R$, define

$$c(a_1, a_2) = \begin{cases} (0, 0) & \text{if } c = 0 \\ (ca_1, \frac{a_2}{c}) & \text{if } c \neq 0. \end{cases}$$

Is V a vector space over R with these operations? Justify your answer.

20. Let V be the set of sequences $\{a_n\}$ of real numbers. (See Example 5 for the definition of a sequence.) For $\{a_n\}, \{b_n\} \in V$ and any real number t , define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \quad \text{and} \quad t\{a_n\} = \{ta_n\}.$$

Prove that, with these operations, V is a vector space over R .

21. Let V and W be vector spaces over a field F . Let

$$Z = \{(v, w) : v \in V \text{ and } w \in W\}.$$

Prove that Z is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c(v_1, w_1) = (cv_1, cw_1).$$

22. How many matrices are there in the vector space $M_{m \times n}(Z_2)$? (See Appendix C.)

1.3 SUBSPACES

In the study of any algebraic structure, it is of interest to examine subsets that possess the same structure as the set under consideration. The appropriate notion of substructure for vector spaces is introduced in this section.

Definition. A subset W of a vector space V over a field F is called a **subspace** of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V .

In any vector space V , note that V and $\{0\}$ are subspaces. The latter is called the **zero subspace** of V .

Fortunately it is not necessary to verify all of the vector space properties to prove that a subset is a subspace. Because properties (VS 1), (VS 2), (VS 5), (VS 6), (VS 7), and (VS 8) hold for all vectors in the vector space, these properties automatically hold for the vectors in any subset. Thus a subset W of a vector space V is a subspace of V if and only if the following four properties hold.

1. $x + y \in W$ whenever $x \in W$ and $y \in W$. (W is **closed under addition**.)
2. $cx \in W$ whenever $c \in F$ and $x \in W$. (W is **closed under scalar multiplication**.)
3. W has a zero vector.
4. Each vector in W has an additive inverse in W .

The next theorem shows that the zero vector of W must be the same as the zero vector of V and that property 4 is redundant.

Theorem 1.3. Let V be a vector space and W a subset of V . Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V .

- (a) $0 \in W$.
- (b) $x + y \in W$ whenever $x \in W$ and $y \in W$.
- (c) $cx \in W$ whenever $c \in F$ and $x \in W$.

Proof. If W is a subspace of V , then W is a vector space with the operations of addition and scalar multiplication defined on V . Hence conditions (b) and (c) hold, and there exists a vector $0' \in W$ such that $x + 0' = x$ for each $x \in W$. But also $x + 0 = x$, and thus $0' = 0$ by Theorem 1.1 (p. 11). So condition (a) holds.

Conversely, if conditions (a), (b), and (c) hold, the discussion preceding this theorem shows that W is a subspace of V if the additive inverse of each vector in W lies in W . But if $x \in W$, then $(-1)x \in W$ by condition (c), and $-x = (-1)x$ by Theorem 1.2 (p. 12). Hence W is a subspace of V . ■

The preceding theorem provides a simple method for determining whether or not a given subset of a vector space is a subspace. Normally, it is this result that is used to prove that a subset is, in fact, a subspace.

The **transpose** A^t of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging the rows with the columns; that is, $(A^t)_{ij} = A_{ji}$. For example,

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ -2 & 5 \\ 3 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

A **symmetric matrix** is a matrix A such that $A^t = A$. For example, the 2×2 matrix displayed above is a symmetric matrix. Clearly, a symmetric matrix must be square. The set W of all symmetric matrices in $M_{n \times n}(F)$ is a subspace of $M_{n \times n}(F)$ since the conditions of Theorem 1.3 hold:

1. The zero matrix is equal to its transpose and hence belongs to W .

It is easily proved that for any matrices A and B and any scalars a and b , $(aA + bB)^t = aA^t + bB^t$. (See Exercise 3.) Using this fact, we show that the set of symmetric matrices is closed under addition and scalar multiplication.