May 3, 2013

Section 6.1

15. (b) (i) We show that $\|x+y\| = \|x\| + \|y\|$ if and only if one of x, y is a *non-negative* multiple of the other.

    ($\Leftarrow$) After rearrangement, we may assume that $x = cy$ for some $c \geq 0$. Then

    $$\|x+y\| = \|(c+1)y\| = |c+1|.\|y\| = (c+1)\|y\| = c\|y\| + \|y\| = \|cy\| + \|y\| = \|x\| + \|y\|.$$

    ($\Rightarrow$) If one of x or y is zero, say $x = 0$, then $x = c.y$ with $c = 0$, which is non-negative. Thus we now assume both x and y are non-zero. We have

    $$\|x+y\|^2 = \|x\|^2 + 2\Re(\langle x, y\rangle) + \|y\|^2$$

    and

    $$(\|x\| + \|y\|)^2 = \|x\|^2 + 2\|x\|.\|y\| + \|y\|^2.$$

    Thus we obtain

    $$\Re(\langle x, y\rangle) = \|x\| \cdot \|y\|. \tag{1}$$

    In particular $\Re(\langle x, y\rangle) \geq 0$. On the other hand,

    $$\begin{aligned} \|x\| \cdot \|y\| &\geq |\langle x, y\rangle| \quad \text{(Cauchy-Schwarz)} \\ &\geq |\Re(\langle x, y\rangle)| \\ &= \|x\| \cdot \|y\| \quad \text{(by (1))}. \end{aligned}$$

    Hence $|\langle x, y\rangle| = \|x\|.\|y\|$ and part (a) then implies that $x = cy$ for some $c \in F$.

    We then have

    $$\Re(\langle x, y\rangle) = \Re(\langle cy, y\rangle) = \Re(c) \cdot \|y\|^2$$

    and

    $$\|x\| \cdot \|y\| = \|cx\| \cdot \|y\| = |c| \cdot \|y\|^2.$$

    Since $\|y\| \neq 0$, the equation (1) says $\Re(c) = |c|$, which implies $c \in \mathbb{R}$ and $c \geq 0$.

    (ii) Consider vectors $x_1, \cdots, x_m \in V$ such that

    $$\|x_1 + \cdots + x_m\| = \|x_1\| + \cdots + \|x_m\|. \tag{2}$$

    We want to show that this is true if and only if all vectors should point to the same direction. There are several ways to say this last condition mathematically. For example, one can use one of the following conditions:

    (A) There exists a non-zero vector $v \in V$ and $\alpha_i \geq 0$ such that $x_i = \alpha_i v$ for all i.

    (B) There exists a vector $v \in V$ with $\|v\| = 1$ and $\alpha_i \geq 0$ such that $x_i = \alpha_i v$ for all i.

    (C) For any $i \neq j$, there exist $a, b \geq 0$, but not both zero, such that $ax_i = bx_j$.

    (D) For any $i \neq j$, there exists $c \geq 0$ such that either $x_i = cx_j$ or $cx_i = x_j$.

    Exercise. Show that (A), (B), (C), (D) are equivalent.

    In the following we show by induction that (2) holds if and only if condition (B) is true.

The direction ($\Leftarrow$) is clear since

$$\begin{aligned}
\|x_1 + \cdots + x_m\| = \left\| \left(\sum_{i=1}^{m} \alpha_i \right) v \right\| &= \left| \sum_{i=1}^{m} \alpha_i \right| \\
&= \sum_{i=1}^{m} \alpha_i \\
&= \sum_{i=1}^{m} \|\alpha_i v\| = \sum_{i=1}^{m} \|x_i\|.
\end{aligned}$$

Consider the direction ($\Rightarrow$). If $m = 1$, we simply

$$\begin{cases} \text{let } v = x_1/\|x_i\| \text{ and } \alpha_1 = \|x_1\| & \text{if } x_1 \neq 0 \\ \text{pick any } v \text{ with } \|v\| = 1 \text{ and } \alpha_1 = 0 & \text{if } x_1 = 0 \end{cases}$$

and we are done.

Assume now $m > 1$ and suppose that if we have $\|\sum_{i=1}^{m-1} x_i\| = \sum_{i=1}^{m-1} \|x_i\|$, then condition (B) holds. Suppose we have m vectors $x_1, \cdots, x_m \in V$ and we can assume that *none of them is zero.* We have

$$\begin{aligned}
\sum_{i=1}^{m} \|x_i\| &= \left\| \sum_{i=1}^{m} x_i \right\| \quad \text{(our assumption)} \\
&= \left\| \left(\sum_{i=1}^{m-1} x_i \right) + x_m \right\| \\
&\leq \left\| \left(\sum_{i=1}^{m-1} x_i \right) \right\| + \|x_m\| \quad \text{(triangle inequality)} \\
&\leq \sum_{i=1}^{m} \|x_i\| \quad \text{(use triangle inequality iteratively).}
\end{aligned}$$

Thus the inequalities above are indeed equalties and we have $\|\sum_{i=1}^{m-1} x_i\| = \sum_{i=1}^{m-1} \|x_i\|$ by the last equality. By induction hypothesis, there exist $v \in V$ with $\|v\| = 1$ and $\alpha_i \geq 0, 1 \leq i < m$ such that $x_i = \alpha_i v$ for $1 \leq i < m$.

On the other hand, if we let $y = \sum_{i=1}^{m-1} x_i$, then one has $\|y + x_m\| = \|y\| + \|x_m\|$ by the third equality above. Thus by part (i), y and x_m point to the same direction. [*Warning:* Here one *cannot* use the induction hypothesis. **Exercise:** why?] Since we already assume that none of x_i is zero, we have $y \neq 0$ and $x_m = cy$ for some $c \geq 0$. Therefore $x_m = \alpha_m v$ for $\alpha_m = c(\alpha_1 + \cdots + \alpha_{m-1}) \geq 0$.

29. Be careful that $[\bullet, \bullet]$ is in $\mathbb{R}$ and hence $[iv, w] \neq i[v, w]$ in general. We first show that

$$[ix, y] = -[x, iy] \tag{3}$$

for any $x, y \in V$. Indeed we have

$$0 = [x + y, i(x + y)] = [x, ix] + [x, iy] + [y, ix] + [y, iy] = [x, iy] + [y, ix].$$

(In fact, for any $c \in \mathbb{C}$, we have $[cx, y] = [x, \bar{c}y]$.)

Use this, it is easy to prove that $\langle \bullet, \bullet \rangle$ is an inner product. For example, let $x, y \in V$ and $c \in \mathbb{C}$. Write $c = a + bi$ with $a, b \in \mathbb{R}$. We have

$$\langle cx, y \rangle = [ax + bix, y] + i[ax + bix, iy] = a[x, y] + b[ix, y] + ia[x, iy] + ib[ix, iy], \tag{4}$$

while

$$c\langle x, y\rangle = (a+bi)\big([x,y] + i[x,iy]\big) = a[x,y] - b[x,iy] + ia[x,iy] + ib[x,y]. \tag{5}$$

Then (4) = (5) by (3).