Section 5.4

20. The direction ($\Leftarrow$) is easy, so we only consider the other direction ($\Rightarrow$).

Suppose that $\dim V = n$ and V is T-generated by a vector $v \in V$. Then the set $\beta = \{v, T(v), \cdots, T^{n-1}(v)\}$ forms a basis of V. There exist $\alpha_0, \cdots, \alpha_{n-1} \in F$ such that

$$U(v) = \alpha_0 v + \cdots + \alpha_{n-1} T^{n-1}(v). \tag{1}$$

Let $g(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_{n-1} t^{n-1}$. We want to show that $U = g(T)$. To do this, we check that U and $g(T)$ send each vector in the basis β to the same image. We have, for $0 \leq j \leq n-1$,

$$\begin{aligned} U(T^j(v)) &= T^j(U(v)) \quad (\text{since } UT = TU) \\ &= T^j(\alpha_0 v + \alpha_1 T(v) + \cdots + \alpha_{n-1} T^{n-1}(v)) \quad (\text{by } (1)) \\ &= (\alpha_0 + \alpha_1 T + \cdots + \alpha_{n-1} T^{n-1})(T^j(v)) \quad (T^i \text{ and } T^j \text{ commute}) \\ &= g(T)(T^j(v)). \end{aligned}$$

Thus $U = g(T)$.

Section 7.2

8(b). (i) We first show that γ' is also a cycle. If we write

$$\gamma = \{\cdots, v, x\},$$

we have $(T - \lambda)(x) = v$ and the vectors in front of v are determined by v via applying powers of $(T - \lambda)$. Now to show that

$$\gamma' = \{\cdots, v, x + y\}$$

is still a cycle, we then only need to show that $(T - \lambda)(x + y) = v$. We have

$$\begin{aligned} (T - \lambda)(x + y) &= (T - \lambda)(x) + (T - \lambda)(y) \\ &= v \quad (\text{since } T(y) = \lambda y). \end{aligned}$$

(ii) To show that after replacing γ by γ' in the original basis β, we still get a basis β' of V, first notice that cycles in β' and in β have the same initial vectors. This shows that vectors in β' are linearly independent (Thm.7.6 in textbook). Now β' and β have the same number of elements. Therefore β' must be a basis of V.

13(b). By induction. To construct a basis of β_1 of $\ker(T)$, the required condition is empty. So we take any basis to be our β_1, and start running induction.

Now suppose we already construct bases $\beta_1, \cdots, \beta_r$ of $\ker(T), \cdots, \ker(T^r)$ with $\beta_1 \subset \cdots \subset \beta_r$. We want to construct a basis β_{r+1} of $\ker(T^{r+1})$ with $\beta_r \subset \beta_{r+1}$. But $\ker(T^r) \subset \ker(T^{r+1})$ so we can regard β_r as a linearly independent subset of $\ker(T^{r+1})$. We then can extend β_r to a basis β_{r+1} of $\ker(T^{r+1})$ and this finishes the induction procedure.

17. The decomposition $T = S + U$ is sometimes called the *semisimple-nilpotent* decomposition. Here semisimple means diagonalizable.

(b) Let β be a Jordan basis of T. To show that U is nilpotent and $SU = US$, we only need to check that $U^p(v) = 0$ for some $p \geq 1$ and $S(U(v)) = U(S(v))$ for each vector v in

β. So take one v in β. It is then in some cycle of generalized eigenvectors of eigenvalue λ. Write this cycle as

$$\{\underbrace{\cdots, v}_{q}, \cdots\}.$$

Then $S(v) = \lambda v$ (v is one eigenvectors of S) by definition and

$$\begin{aligned} U^q(v) &= (T-S)^q(v) \\ &= (T-\lambda)^q(v) \\ &= 0 \quad \text{(by looking at the structure of the dot diagram).} \end{aligned}$$

On the other hand, first notice that $\widetilde{E}_\lambda$ is U-invariant because $U = T - S$ and $\widetilde{E}_\lambda$ is both T- and S-invariant. But S on $\widetilde{E}_\lambda$ is just the multiplication by λ so it commutes with any other operator on $\widetilde{E}_\lambda$. This shows $S(U(v)) = U(S(v))$, which is what we want. (We can also show that $S(U(v)) = U(S(v))$ by using the dot diagram and explicit compuation.)

Section 7.3

9. Since T is diagonalizable, we have

$$\mathrm{ch}_T(x) = \prod_{i=1}^{r}(x-\lambda_i)^{m_i} \quad , \quad \min_T(x) = \prod_{i=1}^{r}(x-\lambda_i)$$

where $\lambda_1, \cdots, \lambda_r$ are distinct eigenvalues of T.

($\Rightarrow$) Since V is T-cyclic, there exists $v \in V$ such that $\{v, T(v), \cdots, T^{n-1}(v)\}$ is a basis of V where $n = \dim V$. Thus if $p(x) \in F[x]$ is of degree $< n$, then $p(T)$ cannot be zero since $p(T)(v) \neq 0$. We therefore obtain that $\deg \min_T(x) = \deg \mathrm{ch}_T(x)$. This can only happen if $\min_T(x) = \mathrm{ch}_T(x)$, which then implies T has n distinct eigenvalues (and each eigenspace must have dimension one). (This is just Thm.7.15 in the textbook.)

($\Leftarrow$) In this case, T has n distinct eigenvalues $\lambda_1, \cdots, \lambda_n$ and

$$\min_T(x) = \mathrm{ch}_T(x) = \prod_{i=1}^{n}(x-\lambda_i).$$

Let v_i be an eigenvector with eigenvalue λ_i for each $i = 1, \cdots, n$. Note that $\beta = \{v_1, \cdots, v_n\}$ is a basis of V. Let $v = v_1 + \cdots + v_n$. We now show that V is T-cyclic generated by v.

Let W be the T-cyclic subspace generated by v. Then W is T-invariant and contains v. The last statement then implies that $v_i \in W$ for each i by Exercise 5.4.(23). (Make sure you know how to prove the exercise.) Thus $W = V$ since $\{v_i\}$ spans V.

In general we have the following.

Exercise. Fix $T \in \mathcal{L}(V)$ and let $\min_T(x)$ be the minimal polynomial of T. Then there exists a T-cyclic subspace W of V such that the characteristic polynomial of T_W is $\min_T(x)$.

Section 7.4

6. (a) The dot diagram for $\widetilde{E}(\phi_1)$ contains only one point. Thus if v corresponds to that point, then $\phi_1(T)(v) = 0$ and $\beta_{v_1} = \{v, T(v), \cdots, T^{d-1}(v)\}$ forms a basis of $\widetilde{E}(\phi_1(x))$ (where $d := \deg \phi_1$). Therefore v_1 has T-annihilator $\phi_1(x)$. The construction for $\phi_2(x)$ is similar.

All together $\beta_{v_1} \cup \beta_{v_2}$ forms a (rational canonical) basis of V.

(b) Let v_1 and v_2 be the vectors constructed in (a). Let $v_3 = v_1 + v_2$. We now show that this v_3 is what we want.

First $\phi_1(T)\phi_2(T)(v_3) = 0$ because $\text{ch}_T(x) = \phi_1(x)\phi_2(x)$ together with the Cayley-Hamilton theorem. Secondly we have

$$\begin{aligned} \phi_1(T)(v_3) &= \phi_1(T)(v_1) + \phi_1(T)(v_2) \\ &= \phi_1(T)(v_2) \quad (\text{since } \phi_1(T)(v_1) = 0). \end{aligned}$$

The last term is not zero because $\phi_1(T)$ is injective on $\widetilde{E}(\phi_2(x))$. Similarly $\phi_2(T)(v_3) \neq 0$. Thus the polynomial $\phi_1(x)\phi_2(x)$ is the smallest monic polynomial which annihilates v_3 and hence $\{v_3, T(v_3), \cdots, T^{n-1}(v_3)\}$ must be linearly independent.

(c) We have

$$[T]_{\beta_{v_1} \cup \beta_{v_2}} = C\big(\phi_1(x)\big) \oplus C\big(\phi_2(x)\big)$$

containing two companion forms, while

$$[T]_{\beta_{v_3}} = C\big(\phi_1(x)\phi_2(x)\big)$$

is a single companion form.