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Section 6.4

17. Notice that T and U are diagonalizable (self-adjoint) and the eigenvalues are non-negative ($T, U \geq 0$ and Exercise (17.a)).

    (c) [Cf. Exercise 13 of §6.6] ($\Rightarrow$) Take an orthonormal basis $\gamma = \{w_1, \cdots, w_n\}$ of V such that $T(w_i) = \lambda_i w_i$. Since $T \geq 0$, we have $\lambda_i \geq 0$. Define a new operator S on V by setting

    $$S(w_i) = \sqrt{\lambda_i} \cdot w_i.$$

    Then S is self-adjoint ($[S^*]_\gamma = ([S]_\gamma)^* = [S]_\gamma$) and $T = S^2 = S^*S$. Thus we have $A = [T]_\beta = B^*B$ where $B = [S]_\beta$.

    ($\Leftarrow$) We have

    $$\begin{aligned} \langle T(v), v\rangle &= \phi_\beta(v)^* \cdot \phi_\beta(T(v)) \\ &= \phi_\beta(v)^* \cdot \big(A \cdot \phi_\beta(v)\big) \\ &= \phi_\beta(v)^* \cdot B^*B \cdot \phi_\beta(v) \\ &= \big(B \cdot \phi_\beta(v)\big)^* \big(B \cdot \phi_\beta(v)\big) \\ &\geq 0. \end{aligned}$$

    [Or one lets $S : V \to V$ be the linear operator such that $[S]_\beta = B$. Then $T = S^*S$ and $\langle T(v), v\rangle = \langle S^*S(v), v\rangle = \langle S(v), S(v)\rangle \geq 0$.]

    (d) First we show that T and U have the same eigenvalues. If $\lambda_1, \cdots, \lambda_n$ and $\mu_1, \cdots, \mu_n$ are the eigenvalues of T and U, respectively, then T^2 and U^2 have eigenvalues λ_i^2 and μ_i^2, respectively. Since $T^2 = U^2$ (our assumption) and $\lambda_i, \mu_i \geq 0$ ($T, U \geq 0$), we see that T and U have the same collection of eigenvalues.

    Fix an eigenvalue λ of T, which is also an eigenvalue of U. Let W_1 and W_2 be the eigenspaces with eigenvalue λ of T and U, respectively. We now show that $W_1 = W_2$. Indeed we have

    $$W_1 = \ker(T^2 - \lambda^2) = \ker(U^2 - \lambda^2) = W_2.$$

    (**Exercise.** Make sure you know (i) $W_1 = \ker(T^2 - \lambda^2)$, and (ii) when self-adjoint T and U have the same eigenvalues and same corresponding eigenspaces, then $T = U$.)

    (e) Since $TU = UT$, they can be diagonalized simultaneously (see Exercise 14 of §6.4). Take an orthonormal basis $\gamma = \{w_1, \cdots, w_n\}$ such that $T(w_i) = \lambda_i w_i$ and $U(w_i) = \mu_i w_i$ for some $\lambda_i, \mu_i > 0$. Then $TU(w_i) = \lambda_i \mu_i w_i$ with eigenvalue $\lambda_i \mu_i > 0$. Thus $TU > 0$.

    Remark. The argument actually shows that if $T, U \geq 0$ and $TU = UT$, then $TU \geq 0$.

Remark. Exercises 21 and 22 tell us that all possible inner products on a fixed finite dimensional vector space V are connected to each other via positive definite operators in $\mathcal{L}(V)$.

22. Another way to find T is as follows. Let $V^* = \mathcal{L}(V, F)$ be the dual space of V. Recall that we have seen in class the bijection

    $$\begin{aligned} V &\to V^* \\ v &\mapsto T_v \end{aligned}$$

    where $T_v(w) := \langle w, v\rangle$. Now for a fixed $x \in V$, the map $y \mapsto \langle y, x\rangle'$ is in V^*. Thus there exists a unique element in V, call it $T(x)$, such that

    $$\langle y, x\rangle' = \langle y, T(x)\rangle \quad \text{(and hence } \langle x, y\rangle' = \langle T(x), y\rangle\text{)} \tag{1}$$

for all $y \in V$. This function T is what we want and we are asked to play a game using the relation (1). Explicitly we need to show that T is (i) linear, (ii) $\langle T(x), y\rangle = \langle x, T(y)\rangle$, (ii') $\langle T(x), y\rangle' = \langle x, T(y)\rangle'$, (iii) $\langle T(x), x\rangle > 0$ for all $x \neq 0$, and (iii') $\langle T(x), x\rangle' > 0$ for all $x \neq 0$.

(i) To check that $T(x_1 + x_2) = T(x_1) + T(x_2)$, observe that for all $y \in V$, we have

$$\begin{aligned}\langle y, T(x_1 + x_2)\rangle = \langle y, x_1 + x_2\rangle' &= \langle y, x_1\rangle' + \langle y, x_2\rangle' \\ &= \langle y, T(x_1)\rangle + \langle y, T(x_2)\rangle = \langle y, T(x_1) + T(x_2)\rangle.\end{aligned}$$

Thus the claim. The rest are left as an exercise.

(ii) We have

$$\langle T(x), y\rangle = \overline{\langle y, T(x)\rangle} = \overline{\langle y, x\rangle'} = \langle x, y\rangle' = \langle x, T(y)\rangle.$$

(ii'), (iii) and (iii') are left as exercises.

23. Following the *Hint* in the textbook, we let $\beta = \{v_i, \cdots, v_n\}$ be a basis of V such that $U(v_i) = \lambda_i v_i$ with $\lambda_i \in \mathbb{R}$. Define a new inner product on V by requiring

$$\langle v_i, v_j\rangle' = \delta_{ij}.$$

Then there exists a positive definite $T_1 \in \mathcal{L}(V)$ such that $\langle x, y\rangle' = \langle T_1(x), y\rangle$ for all $x, y \in V$. The condition $\lambda_i \in \mathbb{R}$ implies that $\langle U(v_i), v_j\rangle' = \langle v_i, U(v_j)\rangle'$, i.e. U is self-adjoint with respect to $\langle \cdot, \cdot\rangle'$.

In the following, we check that $U = T_1^{-1} U^* T_1$.

First we have that T_1^{-1} is also self-adjoint (although we do not really need this). In fact, taking adjoint of $I = T_1^{-1} T_1$, we obtain $I = T_1^*(T_1^{-1})^* = T_1(T_1^{-1})^*$. Hence $(T_1^{-1})^* = T_1^{-1}$.

Now we show that $U(v_i) = T_1^{-1} U^* T_1(v_i)$ by comparing $\langle U(v_i), v_j\rangle$ and $\langle T_1^{-1} U^* T_1(v_i), v_j\rangle$. We have

$$\langle U(v_i), v_j\rangle = \langle \lambda_i v_i, v_j\rangle.$$

On the other hand,

$$\begin{aligned}\langle T_1^{-1} U^* T_1(v_i), v_j\rangle &= \langle U^* T_1(v_i), T_1^{-1}(v_j)\rangle \quad (T_1^{-1} \text{ self-adjoint}) \\ &= \langle T_1(v_i), U T_1^{-1}(v_j)\rangle \\ &= \langle v_i, U T_1^{-1}(v_j)\rangle' \quad (\text{the definition of } T_1) \\ &= \langle U(v_i), T_1^{-1}(v_j)\rangle' \quad (U \text{ self-adjoint w.r.t. } \langle \cdot, \cdot\rangle') \\ &= \langle \lambda_i v_i, T_1^{-1}(v_j)\rangle' \quad (\text{definition of } v_i) \\ &= \langle T_1(\lambda_i v_i), T_1^{-1}(v_j)\rangle \quad (\text{why?}) \\ &= \langle T_1^{-1} T_1(\lambda_i v_i), v_j\rangle \quad (\text{why?}) \\ &= \langle \lambda_i v_i, v_j\rangle.\end{aligned}$$

Thus the assertion follows.

Section 6.5

15. Notice that the arguments below do not require that V is finite dimensional.

(0) U is one-to-one. Indeed if $U(v) = 0$ for some $v \in V$, then $0 = \langle U(v), U(v)\rangle = \langle v, v\rangle$ and hence $v = 0$.

(a) Since U_W is one-to-one and W is finite dimensional, the inclusion $U(W) \subset W$ implies that $U(W) = W$.

(b) Let $v \in W^\perp$. We want to show that $\langle U(v), w\rangle = 0$ for any $w \in W$. By (a), there exists $w' \in W$ such that $U(w') = w$. Thus $\langle U(v), w\rangle = \langle U(v), U(w')\rangle = \langle v.w'\rangle = 0$.

Section 6.6

5. (b) Since T is a projection, we have

$$\ker(T) = E_0, \quad \operatorname{Im}(T) = E_1, \quad V = E_0 \oplus E_1.$$

Now under the condition $\|T(x)\| \leq \|x\|$ for all $x \in V$, we want to show that $E_1 = E_0^{\perp}$. We first show that $E_1 \subset E_0^{\perp}$. Consider the new decomposition

$$V = E_0 \oplus E_0^{\perp}.$$

Let $v \in E_1$ and write $v = w + w'$ where $w \in E_0$ and $w' \in E_0^{\perp}$. We have

$$\|v\|^2 = \|w\|^2 + \|w'\|^2 \tag{2}$$

since $w \perp w'$. On the other hand, we have

$$v = T(v) = T(w + w') = T(w')$$

and we obtain

$$\|v\| = \|T(w')\| \leq \|w'\|. \tag{3}$$

Combining (2) and (3), we see that $\|w\| = 0$. Hence $w = 0$ and we obtain that $v \in E_0^{\perp}$. This implies that $E_1 \subset E_0^{\perp}$.

Since $\dim E_1 = \dim V - \dim E_0 = \dim E_0^{\perp}$, one obtains $E_1 = E_0^{\perp}$.

Section 6.7

9. Let $\beta = \{v_1, \cdots, v_n\}$ and $\gamma = \{u_1, \cdots, u_m\}$. Notice that $[T^*]_{\gamma}^{\beta} = \left([T]_{\beta}^{\gamma}\right)^*$ by Exercise 15 of §6.3, which we have discussed in class. Thus $T^*(u_j) = \sigma_j v_j$.