Section 5.2

23. We have that

$$\sum_{i=1}^{p} K_i + \sum_{j=1}^{q} M_j = W_1 + W_2 = W_1 \oplus W_2. \tag{1}$$

On the other hand, let β_i and γ_j be bases of K_i and M_j, respectively. Since $W_1 = \bigoplus_{i=1}^{p} K_i$, the (disjoint) union β of β_i is a basis of W_1. Similarly the union γ of γ_j is a basis of W_2 and the union of β and γ forms a basis of $W_1 \oplus W_2$. Thus the disjoint union of β_i and γ_j is a basis of $W_1 \oplus W_2$ and this shows that the sum in the left hand side of (1) is a direct sum by Theorem 5.10(d).

Section 7.1

9(b). Clearly β' is a linearly independent set in K_λ. To show that β' is a basis of K_λ, we shall show that β' consists of at least $\dim K_\lambda$ elements,

The set β is a Jordan canonical basis for T means that the matrix $[T]_\beta$ is the Jordan canonical form of T. According to the sizes of Jordan blocks in $[T]_\beta$, the basis β decomposes into cycles $\gamma_1, \cdots, \gamma_r$ where each cycle γ_i corresponds to a Jordan block with diagonal entries equal to λ_i. (Thus λ_i are among the eigenvalues of T and some of them might be the same.) Let ℓ_i be the length of γ_i, which is equal to the size of the corresponding Jordan block. Then

$$\mathrm{ch}_T(x) = \det\left(xI - [T]_\beta\right) = \prod_{i=1}^{r} (x - \lambda_i)^{\ell_i}.$$

This implies that

$$\dim K_\lambda = \sum_{\lambda_i = \lambda} \ell_i.$$

On the other hand, those γ_i with $\lambda_i = \lambda$ are contained in K_λ because $(T-\lambda)^{\ell_i}(\gamma_i) = 0$. This shows that β' contains at least $\sum_{\lambda_i = \lambda} \ell_i$ elements, which is what we want.

16. First notice that by deleting the last i dots in each column, the resulting set β' is contained in $\mathrm{Im}(T^i)$ because each remaining dot (= a vector in β') is mapped by another dot in the same column by moving up the dots i steps. Let $\ell_1, \cdots, \ell_r$ be the lengths of the columns in the dot diagram. Then

$$\#\beta' = \sum_{i=1}^{r} (\ell_k - i)_+ \quad \text{where} \quad (\ell_k - i)_+ := \begin{cases} \ell_k - i & \text{if } \ell_k - i \geq 0 \\ 0 & \text{if } \ell_k - i \leq 0. \end{cases}$$

On the other hand, $[T]_\beta$ consists of r Jordan blocks of sizes $\ell_1, \cdots, \ell_r$, and the diagonal entries are all zero. By direct computation, $[T]_\beta^i$ has $\sum_{i=1}^{r} (\ell_k - i)_+$ linearly independent columns (see the computation in Exercise 19(a)). Thus

$$\mathrm{rk}(T^i) = \mathrm{rk}([T]_\beta^i) = \#\beta',$$

which implies that β' is a basis of $\mathrm{Im}(T^i)$.

(Or one can use that the dots you delete = the total dots obtained by adding first i dots in each column = $\nu(T^i)$. Thus $\#\beta' = \dim V - \nu(T^i) = \mathrm{rk}(T^i)$.)