There are eight problems 1 $\sim$ 8 in total; some problems contain sub-problems, indexed by (a), (b), etc.

In the following, all vector spaces are assumed to be finite dimensional. the characteristic polynomial of a square matrix $A \in M_n(F)$ is defined to be $\det(xI - A)$. Similarly for $T \in \mathcal{L}(V)$, the characteristic polynomial of T is defined by

$$\det(xI - [T]_\beta)$$

where $[T]_\beta$ is the matrix representation of T with respect to any ordered basis β of V.

1. [20%] True or false. (No explanation is needed.)
   (a) Any square matrix $A \in M_n(F)$ is similar to a Jordan form $J \in M_n(F)$.
   (b) $T \in \mathcal{L}(T)$ is diagonalizable if and only if the minimal polynomial of T splits.
   (c) Let $T \in \mathcal{L}(V)$. Let W_1 and W_2 be two T-cyclic subspaces of V generated by w_1 and w_2, respectively. Suppose $W_1 = W_2$. Then $w_1 = w_2$.
   (d) Let $T \in \mathcal{L}(V)$ and $v \in V$. The T-cyclic subspaces generated by v and $T(v)$ are the same.
   (e) Let $T \in \mathcal{L}(V)$. Suppose W_1 and W_2 are T-invariant subspaces of V and $V = W_1 \oplus W_2$. Then the characteristic polynomial of T is the product of characteristic polynomials of T_{W_1} and of T_{W_2}.
   (f) Let $T \in \mathcal{L}(V)$. Suppose W_1 and W_2 are T-invariant subspaces of V and $V = W_1 \oplus W_2$. Then the minimal polynomial of T is the product of minimal polynomials of T_{W_1} and of T_{W_2}.
   (g) Let $T \in \mathcal{L}(V)$ and $\widetilde{E}_\lambda$ be the generalized eigenspace of V with eigenvalue λ. Then $\widetilde{E}_\lambda$ has a basis which is a union of cycles of generalized eigenvectors.
   (h) Let $T \in \mathcal{L}(V)$. Then V is a direct sum of T-cyclic subspaces.

2. [15%] Let $T \in \mathcal{L}(V)$ with Jordan canonical form

$$A := J_2(2) \oplus J_2(2) \oplus J_3(3) = \begin{pmatrix} 2 & 1 & & & & & \\ & 2 & & & & & \\ & & 2 & 1 & & & \\ & & & 2 & & & \\ & & & & 3 & 1 & \\ & & & & & 3 & 1 \\ & & & & & & 3 \end{pmatrix}.$$

   (a) Find the characteristic polynomial and the minimal polynomial of T.
   (b) Find the nullities of the following 6 linear transformations

$$T - 2I, \quad (T - 2I)^2, \quad (T - 2I)^3,$$
$$T - 3I, \quad (T - 3I)^2, \quad (T - 3I)^3.$$

   (c) Now suppose the field is $\mathbb{R}$. Find the exponential e^A explicitly.

3. [15%] Let $T \in \mathcal{L}(V)$. Let $p(x)$ be the characteristic polynomial and $q(x)$ the minimal polynomial of T. Prove the following results.
   (a) There exists a positive integer r such that $p(x)$ divides $q(x)^r$.
   (b) T is invertible if and only if $q(0) \neq 0$.

4. [10%] Let $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix} \in M_4(\mathbb{R})$ whose characteristic polynomial is

$$x^4 - 4x^3 + 14x^2 - 20x + 25 = (x^2 - 2x + 5)^2.$$

Find an invertible matrix P and the rational canonical form Q of A such that $P^{-1}AP = Q$. (Notice that $A^2 - 2A + 5I = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.)

5. [20%] Let $S, T \in \mathcal{L}(V)$. Suppose $ST = 0 = TS$ and $S + T = I$. Show that
   (a) $V = \ker(S) \oplus \ker(T)$ and $V = \operatorname{Im}(S) \oplus \operatorname{Im}(T)$.
   (b) $\ker(S) = \operatorname{Im}(T)$ and $\operatorname{Im}(S) = \ker(T)$.

6. [15%] Prove that any square matrix $A \in M_n(F)$ is similar to its transport A^t. (You may use Jordan forms to get partial credits.)

7. [15%] Suppose $\dim V = n$ and $T \in \mathcal{L}(V)$ is diagonalizable. Show that V is T-cyclic if and only if T has n distinct eigenvalues.

8. [30%] Let V be a finite dimensional vector space over the field F and $T \in \mathcal{L}(V)$.
   (a) Let k be the degree of the minimal polynomial of T. Show that there exists a vector $v \in V$ such that the T-cyclic subspace $W := \langle v, T(v), T^2(v), \cdots \rangle$ has dimension k.
   (b) Prove the following version of rational canonical forms. There exist polynomials $f_1(x), f_2(x), \cdots, f_r(x) \in F[x]$ with

$$f_1(x) \big| f_2(x), \quad f_2(x) \big| f_3(x), \quad \cdots, \quad f_{r-1}(x) \big| f_r(x)$$

   and an ordered basis β of V such that

$$[T]_\beta = C\big(f_1(x)\big) \oplus C\big(f_2(x)\big) \oplus \cdots \oplus C\big(f_r(x)\big).$$

   Here $f(x) \big| g(x)$ means $f(x)$ divides $g(x)$, and $C\big(f_i(x)\big)$ denotes the companion form.