There are eight problems 1 $\sim$ 8 in total; some problems contain sub-problems, indexed by (a), (b), etc.

In the following, all vector spaces are assumed to be finite dimensional. For any $A = (a_{ij}) \in M_n(\mathbb{C})$, the adjoint $A^* = (b_{ij})$ of A is defines by letting $b_{ij} = \overline{a_{ji}}$.

If $(V, \langle\,,\rangle)$ is an inner product space, the *norm* $\|x\|$ of $x \in V$ is defined by $\sqrt{\langle x, x\rangle}$; denote $x \perp y$ if $\langle x, y\rangle = 0$; for a subset S of V, define $S^\perp = \{v \in V \mid \langle v, s\rangle = 0\ \forall s \in S\}$.

1. [15%] Let V be an inner product space. Prove the following.

   (a) For any $x, y \in V$, we have
   $$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

   (b) Let $x, y \in V$. Then $|\langle x, y\rangle| = \|x\| \cdot \|y\|$ if and only if one of x or y is a multiple of the other.

2. [30%] Let V be an inner product space and W_1 and W_2 be two subspaces of V. Show that

   (a) $(W_1^\perp)^\perp = W_1$;

   (b) $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$;

   (c) $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$.

3. [10%] Let V be an inner product space and $T \in \mathcal{L}(V)$. Prove that a subspace W of V is T-invariant if and only if $W^\perp$ is T^*-invariant. (Here T^* denotes the adjoint of T.)

4. [10%] Find an orthogonal matrix whose first row is $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.

5. [20%] Recall that two square matrices $A, B \in M_n(\mathbb{C})$ are called *unitarily similar* if there exists an unitary $Q \in M_n(\mathbb{C})$ such that $Q^*AQ = B$.

   (a) Suppose $A, B \in M_n(\mathbb{C})$ are normal. Show that A and B are unitarily similar if and only if A and B have the same characteristic polynomial.

   (b) Let $A \in M_n(\mathbb{C})$. Show that A^*A and AA^* are unitarily similar.

6. [10%] Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Suppose $A^r = I$ for some positive integer r. Show that $A^2 = I$.

7. [20%] Let V be a vector space over F and $H : V \times V \to F$ a bilinear form. Suppose H is *alternating* (i.e., $H(v, v) = 0$ for all $v \in V$).

   (a) Show that $H(v, w) = -H(w, v)$ for all $v, w \in V$.

   (b) Show that there exists a basis
   $$\beta = \{v_1, w_1, \cdots, v_a, w_a, u_1, \cdots, u_b\}$$
   of V such that
   $$H(v_i, w_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$
   and
   $$H(v_i, v_j) = H(w_i, w_j) = H(u_i, u_j) = H(u_i, v_j) = H(u_i, w_j) = 0.$$
   (That is, the matrix representation $\psi_\beta(H)$ is a direct sum of some copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ corresponding to $\{v_i, w_i\}$ and copies of zeros corresponding to $\{u_j\}$.)

8. [30%] Recall that a symmetric matrix $A \in M_n(\mathbb{R})$ is called *positive definite* if $v^t A v > 0$ for any non-zero column vector $v \in \mathbb{R}^n$; it is called *positive semi-definite* if $v^t A v \geq 0$ for all $v \in \mathbb{R}^n$.

Now let $A = (a_{ij}) \in M_n(\mathbb{R})$ be symmetric. Define $A_r = (a_{ij})_{1 \leq i,j \leq r} \in M_r(\mathbb{R})$ and $\Delta_r = \det A_r$ for $1 \leq r \leq n$.

(a) Suppose that $\Delta_r \neq 0$ for all $r = 1, 2, \cdots, n$. Show that there exist independent column vectors $v_1, \cdots, v_n \in \mathbb{R}^n$ such that

$$v_i^t A v_j = \begin{cases} \Delta_1 & i = j = 1 \\ \Delta_r / \Delta_{r-1} & i = j = r > 1 \\ 0 & i \neq j. \end{cases}$$

(If you don't know how to solve this, try the case $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ for partial credit.)

(b) Show that A is positive definite if and only if $\Delta_r > 0$ for all $1 \leq r \leq n$. (You should not use (a) if you don't know how to prove it. Again try $n = 2$ case for partial credit.)

(c) Give an explicit example of symmetric $A \in M_n(\mathbb{R})$ such that $\Delta_r \geq 0$ for all $1 \leq r \leq n$ but A is not positive semi-definite. Remember to justify your answer.