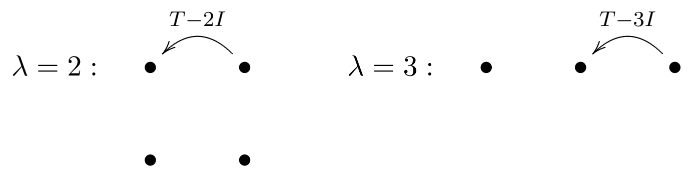
April 17, 2013

1. (a) (F) If $\mathrm{ch}_A(x)$ does not split, A is not similar to a Jordan form.
   (b) (F) For example if $\min_A(x) = x^2$, then A is not diagonalizable.
   (c) (F)
   (d) (F) For example if $v \neq 0$ but $T(v) = 0$.
   (e) (T)
   (f) (F) For example if $\min_{T_{W_1}}(x) = \min_{T_{W_2}}(x)$, then $\min_T(x) = \min_{T_{W_1}}(x)$.
   (g) (T) This follows from the construction of Jordan forms.
   (h) (T) This can be seen from the construction of rational canonical forms.

2. (a)

$$\begin{aligned} \mathrm{ch}_T(x) &= \det(xI - A) = (x-2)^4(x-3)^3 \\ \min_T(x) &= (x-2)^2(x-3)^3. \end{aligned}$$

   (b) The dot diagram of T looks like



We have

$$\begin{aligned} \nu(T-2I) &= \dim E_2 = 2 \\ \nu\left((T-2I)^2\right) &= \dim \widetilde{E}_2 = 4 \\ \nu\left((T-2I)^3\right) &= \dim \widetilde{E}_2 = 4 \\ \nu(T-3I) &= \dim E_3 = 1 \\ \nu\left((T-3I)^2\right) &= 2 \\ \nu\left((T-3I)^3\right) &= \dim \widetilde{E}_3 = 3. \end{aligned}$$

   (c)

$$\begin{aligned} \exp(J_2(2)) = \exp\left(\begin{pmatrix} 2 & \\ & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}\right) &= \exp\left(\begin{pmatrix} 2 & \\ & 2 \end{pmatrix}\right)\exp\left(\begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}\right) \\ &= e^2 I\left[I + \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}\right] \\ &= e^2\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \exp(J_3(3)) &= \exp\left(\begin{pmatrix} 3 & & \\ & 3 & \\ & & 3 \end{pmatrix}\right)\exp\left(\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}\right) \\ &= e^3 I\left[I + \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}\right] \\ &= e^3\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ & 1 & 1 \\ & & 1 \end{pmatrix} \end{aligned}$$

Thus

$$e^A = e^2 \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \oplus e^2 \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \oplus e^3 \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ & 1 & 1 \\ & & 1 \end{pmatrix}.$$

3. (a) If we write

$$p(x) = \phi_1(x)^{m_1} \cdots \phi_r(x)^{m_r}$$

where $\phi_i(x)$ are distinct irreducible factors of $p(x)$, then we have

$$q(x) = \phi_1(x)^{a_1} \cdots \phi_r(x)^{a_r}$$

for some integers a_i with $1 \leq a_i \leq m_i$. Then it is clear that $p(x)\big|q(x)^r$ if we take, for example, $r = \max\{m_1, \cdots, m_r\}$.

(b) $q(0) \neq 0$ means that x is not a factor of $q(x)$. By (a), we know that x is not a factor of $p(x)$ either. This means that if we write

$$p(x) = x^n + \cdots + a_1 x + a_0,$$

then $a_0 \neq 0$. On the other hand, $a_0 = (-1)^n \det(T)$. Thus $\det(T) \neq 0$, which implies that T is invertible.

Or we can prove this directly by showing that T is injective as follows. Write

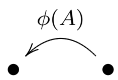
$$q(x) = x^m + \cdots b_1 x + b_0 \quad \text{with } b_0 \neq 0$$

and suppose $T(v) = 0$ for some $v \in V$. Then

$$\begin{aligned} 0 = q(T)(v) &= T^m(v) + \cdots + b_1 T(v) + b_0 v \\ &= (T^{m-1} + \cdots + b_1)(T(v)) + b_0 v \\ &= b_0 v. \end{aligned}$$

Since $b_0 \neq 0$, we have $v = 0$.

4. There is only one irreducible factor $\phi(x) := x^2 - 2x + 5$ of $\mathrm{ch}_A(x) = \phi(x)^2$. Thus the dot diagram consists of two points. Because $\phi(A) \neq 0$, the dot diagram is



and the rational canonical form is

$$Q = C(\mathrm{ch}_A(x)) = \begin{pmatrix} 0 & 0 & 0 & -25 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 4 \end{pmatrix}.$$

To find a cyclic basis corresponding to this digram, we need to find a possible end vector. In this case, this means to find a vector in $\mathbb{R}^4$ which is not in the kernel of $\phi(A)$. We have $\ker(\phi(A)) = \langle e_1, e_2 \rangle$. Thus we can take e_4 to be the end vector. Then

$$P = \Big(e_4, Ae_4, A(Ae_4), A\big(A(Ae_4)\big)\Big) = \begin{pmatrix} 0 & 0 & 4 & 12 \\ 0 & 1 & 2 & -9 \\ 0 & 2 & 4 & -2 \\ 1 & 1 & -3 & -11 \end{pmatrix}.$$

5. Let $\dim V = n$. From $S + T = I$, we have $V = \mathrm{Im}(S) + \mathrm{Im}(T)$ and hence

$$n \le \mathrm{rk}(S) + \mathrm{rk}(T). \tag{1}$$

Since $ST = TS = 0$, we have $\mathrm{Im}(S) \subset \ker(T), \mathrm{Im}(T) \subset \ker(S)$ and in particular,

$$\mathrm{rk}(S) \le \nu(T), \quad \mathrm{rk}(T) \le \nu(S). \tag{2}$$

Together with the dimension formula, we obtain

$$n \underset{(1)}{\le} \mathrm{rk}(S) + \mathrm{rk}(T) \underset{(2)}{\le} \nu(T) + \mathrm{rk}(T) = n.$$

Thus the above two inequalities are indeed equalities and we have

$$\mathrm{Im}(S) = \ker(T), \quad V = \mathrm{Im}(S) \oplus \mathrm{Im}(T).$$

Similarly we have

$$\mathrm{Im}(T) = \ker(S)$$

and hence $V = \ker(S) \oplus \ker(T)$.

6. To show that A and A^t are similar, it suffices to show that they have the same dot diagram, and hence they have the same rational canonical form.

Write

$$\mathrm{ch}_A(x) = \phi_1(x)^{m_1} \cdots \phi_r(x)^{m_r}$$

where $\phi_i(x)$ are distinct irreducible factors of the characteristic polynomial. Then

$$\mathrm{ch}_{A^t}(x) = \phi_1(x)^{m_1} \cdots \phi_r(x)^{m_r}$$

and for any positive integer k, we have

$$\mathrm{rk}\left(\phi_i(A)^k\right) = \mathrm{rk}\left((\phi_i(A)^k)^t\right) = \mathrm{rk}\left(\phi_i(A^t)^k\right).$$

The statement now follows.

7. *Exercise.* Check out Theorem 7.15 in textbook and see why the theorem is not enough to prove this question easily.

($\Longrightarrow$) Since T is diagonalizable,

$$\min_T(x) = \prod_{i=1}^{r}(x - \lambda_i)$$

where λ_i are distinct eigenvalues of T. Now V being T-cyclic implies that $\mathrm{ch}_T(x) = \min_T(x)$. Therefore $r = \dim V$ and each eigenspace is 1-dimensional.

($\Longleftarrow$) Let $n = \dim V$. Take eigenvectors $v_i, 1 \le i \le n$, of T with distinct eigenvalues. Then $\{v_i\}_{i=1}^n$ forms a basis of V. Let $v = v_1 + \cdots + v_n$. We shall show that β_v is a basis of V and hence V is T-cyclic. Indeed the subspace W generated by β_v is the smallest T-invariant subspace containing v. Since v_i are in different eigenspaces, $v \in W$ then implies $v_i \in W$ for all i. Therefore $W = V$.

8. First we prove the following statement.

Let $u, v \in V$ and $w = u + v$. Let $f(x), g(x), h(x) \in F[x]$ be the monic polynomials of smallest possible degree such that $f(T)(u) = g(T)(v) = h(T)(w) = 0$ (i.e. $f(x)$ is the T-annihilator of v and so on). Suppose that $f(x)$ and $g(x)$ are co-prime to each other. Then $h(x) = f(x)g(x)$.

Proof. First notice that $f(T)g(T)(w) = 0$.

Now suppose that $\phi(x)$ is irreducible with $\deg \phi(x) \geq 1$ such that $\phi(x) \mid f(x)g(x)$. Then $\phi(x)$ must divide exact one of $f(x)$ and $g(x)$ because $\gcd(f(x), g(x)) = 1$. After rearrangement, we may assume that $\phi(x) \mid f(x)$ but $\phi(x) \nmid g(x)$. Let $k(x) = f(x)/\phi(x)$, which is then a polynomial. We claim that $k(T)g(T)(w) \neq 0$.

Suppose otherwise that $k(T)g(T)(w) = 0$. Since $w = u + v$ and $g(T)(v) = 0$, we have $k(T)g(T)(u) = 0$. This implies that $f(x) \mid k(x)g(x)$, which is impossible. (Remember that $\gcd(f(x), g(x)) = 1$ and $\deg k(x) < \deg f(x)$.)

Therefore any monic factor $\ell(x)$ of $f(x)g(x)$ of smaller degree will have $\ell(T)(w) \neq 0$. Hence the T-annihilator of w is $f(x)g(x)$. □

(a) Write

$$\mathrm{ch}_T(x) = \phi_1(x)^{m_1} \cdots \phi_k(x)^{m_k} \quad \text{and} \quad \mathrm{min}_T(x) = \phi_1(x)^{a_1} \cdots \phi_k(x)^{a_k}$$

for some $1 \leq a_i \leq m_i$ where $\phi_i(x)$ are distinct monic factors of $\mathrm{ch}_T(x)$. Then there are vectors v_i whose T-annihilators are $\phi_i(x)^{a_i}$. According to the property proved above, we see that $v_1 + v_2$ has T-annihilator $\phi_1(x)^{a_1}\phi_2(x)^{a_2}$; $(v_1 + v_2) + v_3$ has T-annihilator $(\phi_1(x)^{a_1}\phi_2(x)^{a_2})\phi_3(x)^{a_3}$, and inductively $w := \sum_{i=1}^k v_i$ has T-annihilator $\mathrm{min}_T(x)$. Thus w generates a T-cyclic subspace of dimension $= \deg \mathrm{min}_T(x)$.

(b) Let r_i be the number of $\phi_i(T)$-cycles, listed from longer to shorter, in the dot diagram for $\widetilde{E}(\phi_i)$ and let $r = \max\{r_1, r_2, \cdots, r_k\}$. Let $v_{i1}, v_{i2}, \cdots, v_{ir}$ be the end vectors of these cycles of lengths $\ell_{i1} \geq \ell_{i2} \geq \cdots \geq \ell_{ir}$, respectively. (In case $r_i < r$ and there is no such cycle in the dot diagram, we simply take the end vector to be 0.) Then as in (a), the vector

$$w_j := \sum_{i=1}^{k} v_{ij} \tag{3}$$

has T-annihilator $g_j(x) := \prod_{i=1}^k \phi_i(x)^{\ell_{ij}}$ for $j = 1, 2, \cdots, r$, satisfying

$$g_r(x) \mid g_{r-1}(x) \mid \cdots \mid g_2(x) \mid g_1(x).$$

Thus β_{w_j} is a T-cyclic basis of a T-invariant subspace W_j and we have $\left[T_{W_j}\right]_{\beta_{w_j}} = C\big(g_j(x)\big)$.

If now the disjoint union of β_{w_j} forms a basis of V, then the ordered basis $\beta = \beta_{w_r} \cup \cdots \cup \beta_{w_1}$ is what we want (with $f_j = g_{r+1-j}$). So we are left to show that β is a basis of V.

Indeed by the definition (3), it is clear that

$$W_j = \langle \beta_{w_j} \rangle \subset W'_j := \langle \beta_{v_{1j}} \rangle \oplus \langle \beta_{v_{2j}} \rangle \oplus \cdots \oplus \langle \beta_{v_{rj}} \rangle.$$

(The W'_j is the sum of T-cyclic subspaces corresponding to certain cycles in the dot diagram.) However both spaces have dimension $\deg g_j = \sum_{i=1}^k \ell_{ij}$. Thus $W_j = W'_j$ and

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_r.$$

The last equality shows that β is a basis of V.