There are eight problems $1 \sim 8$ in total; some problems contain sub-problems, indexed by (a), (b), etc.

1. [15%] Write down the matrix representations of the following linear transformations with respect to the standard ordered bases.
   (a) The projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ of $\mathbb{R}^3$ onto the yz-coordinate plane.
   (b) The rotation $Q : \mathbb{R}^2 \to \mathbb{R}^2$ of $60°$ on $\mathbb{R}^2$.
   (c) The reflection $R : \mathbb{R}^2 \to \mathbb{R}^2$ along the line $x + 3y = 0$ on $\mathbb{R}^2$.

2. [10%] Let $S, T \in \mathcal{L}(V, W)$ be two non-zero linear maps.
   (a) Prove that if S and T are linearly dependent, then $\mathrm{Im}(S) = \mathrm{Im}(T)$ and $\ker(S) = \ker(T)$.
   (b) Show that if $\mathrm{Im}(S) \cap \mathrm{Im}(T) = \{0\}$, then S and T are linearly independent.

3. [10%] Let $A \in M_{m\times n}(F)$ be an $m \times n$ matrix. Prove that A is of rank r if and only if there exist two matrices $B \in M_{m\times r}(F)$ and $C \in M_{r\times n}(F)$, both of rank r, such that $A = BC$.

4. [15%] Let $T : V \to W$ be a linear map between two vector spaces V and W of dimension n and m, respectively. Suppose $\mathrm{rk}(T) = r$. Show that there exist ordered bases β and γ of V and W, respectively, such that

$$[T]_\beta^\gamma = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

   where $I_r \in M_r(F)$ is the identity matrix and $O_1 \in M_{r,n-r}(F), O_2 \in M_{m-r,r}(F), O_3 \in M_{m-r,n-r}(F)$ are zero matrices.

5. [10%]
   (a) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any $A \in M_{m\times n}(F)$ and $B \in M_{n\times m}(F)$. (Here tr denotes the trace which takes a square matrix to the sum of its diagonal entries.)
   (b) Fix a finite dimensional vector space V. For any $T \in \mathcal{L}(V)$ and an ordered basis β of V, we define the trace $\mathrm{tr}(T)$ of T by

$$\mathrm{tr}(T) := \mathrm{tr}([T]_\beta)$$

   where the right hand side is the usual trace of a matrix. Prove that $\mathrm{tr}(T)$ does not depend on the choice of the ordered basis β.

6. [20%] Let V, W be finite dimensional vector spaces and $T, U \in \mathcal{L}(V, W)$.
   (a) Show that $\mathrm{Im}(T + U) \subset \mathrm{Im}(T) + \mathrm{Im}(U)$.
   (b) Show that $\mathrm{rk}(T + U) \le \mathrm{rk}(T) + \mathrm{rk}(U)$.
   (c) Give an example of V, W, T, U such that $\mathrm{rk}(T + U) = \mathrm{rk}(T) + \mathrm{rk}(U)$.
   (d) Give an example of V, W, T, U such that $\mathrm{rk}(T + U) < \mathrm{rk}(T) + \mathrm{rk}(U)$.

7. [20%] Let V be a finite dimensional vector space and $T \in \mathcal{L}(V)$. Suppose that $T^2 = T$. (Here $T^2(v) := T \circ T(v) = T(T(v))$ for any $v \in V$.) Let $\mathbf{1}_V \in \mathcal{L}(V)$ be the identity transformation and let $S = \mathbf{1}_V - T$.
   (a) Show that $S^2 = S$.
   (b) Show that $\ker(S) = \mathrm{Im}(T)$ and $\mathrm{Im}(S) = \ker(T)$.

(c) Show that $V = \ker(T) \oplus \ker(S)$, i.e., $V = \ker(T) + \ker(S)$ and $\ker(T) \cap \ker(S) = \{0\}$.

(d) Show that $V = \mathrm{Im}(T) \oplus \mathrm{Im}(S)$.

(e) Show that there exists an ordered basis β of V such that $[T]_\beta$ is a diagonal matrix with diagonal entries equal to either 0 or 1.

8. [10%] Compute the determinant of the matrix $A \in M_n(F)$ in terms of $\lambda, a_0, \cdots, a_{n-1}$ where

$$A = \begin{pmatrix} \lambda & 0 & 0 & \dots & 0 & a_0 \\ -1 & \lambda & 0 & \dots & 0 & a_1 \\ 0 & -1 & \lambda & \dots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & a_{n-2} \\ 0 & 0 & 0 & \dots & -1 & \lambda + a_{n-1} \end{pmatrix}$$

(the diagonal entries are λ except the last one, the entries below λ are all -1, the last column is the vector

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \lambda \end{pmatrix} + \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{pmatrix},$$

and other entries are all zero). You may try the cases $n = 2, 3$ and guess the answer first.