(I) (1) (F) $\dim M_{m\times n} = mn$.

(2) (T) In fact, one can take the zero map $T(v) = 0$.

(3) (T) This can be seen from the dimension formula $\nu(T) + rk(T) = \dim V$.

(4) (T) The condition $\ker(T) = \{0\}$ implies that T is 1-1.

(5) (F) $(3, 6) = 3 \cdot (1, 2)$ but $(1, 3, 5) \neq 3 \cdot (2, 4, 6)$.

(6) (F) For example, take V to be a non-zero vector space, T the zero map and $S = \{v\}$ where $v \in V$ is a non-zero vector. In this case S is not a subspace of V but $T(S) = \{0\}$ is a subspace of W.

(7) (T)

(8) (T)

(II) (1) Take $V = \mathbb{R}^2$, $W_1 = x$-axis, $W_2 = y$-axis. Then $W_1 \cup W_2$ is a union of two lines. We have $e_1 = (1, 0) \in W_1, e_2 = (0, 1) \in W_2$, but $e_1 + e_2 = (1, 1) \notin W_1 \cup W_2$.

(2) We show that if $W_1 \not\supset W_2$, then $W_1 \subset W_2$. (Or one can assume that $W_1 \not\subset W_2, W_2 \not\subset W_1$, and derive a contradiction.)

Suppose that $W_1 \not\supset W_2$. Then there exists a vector $v \in W_2 \setminus W_1$. Let $w \in W_1$. We want to show that $w \in W_2$.

Since $v, w \in W_1 \cup W_2$ and $W_1 \cup W_2$ is a subspace, we have $v + w \in W_1 \cup W_2$. Since $v \notin W_1$ and $w \in W_1$, the vector $v + w$ cannot be in W_1. Thus $v + w \in W_2$. The condition $v \in W_2$ then implies that $w \in W_2$, which is what we need.

(III) Suppose that $\alpha \cdot \cos x + \beta \cdot \sin x = 0$ for some $\alpha, \beta \in \mathbb{R}$. Plugging in $x = 0$, we get $\alpha = 0$; plugging in $x = \pi/2$, get $\beta = 0$. Thus $\cos x$ and $\sin x$ are linearly independent.

(IV) (1) Let

$$r = \dim W_1 \cap W_2, \quad r + s = \dim W_1, \quad r + t = \dim W_2.$$

We want to show that

$$\dim(W_1 + W_2) = r + s + t.$$

Let $\{u_1, \cdots, u_r\}$ be a basis of $W_1 \cap W_2$. We extend it to a basis $\{u_1, \cdots, u_r, v_1, \cdots, v_s\}$ of W_1, and to a basis $\{u_1, \cdots, u_r, w_1, \cdots, w_t\}$ of W_2. We now show that the set

$$S = \{u_1, \cdots, u_r, v_1, \cdots, v_s, w_1, \cdots, w_t\}$$

forms a basis of $W_1 + W_2$. For any $a + b \in W_1 + W_2$ where $a \in W_1, b \in W_2$, we can write

$$\begin{aligned} a &= \alpha_1 u_1 + \cdots + \alpha_r u_r + \beta_1 v_1 + \cdots + \beta_s v_s \\ b &= \gamma_1 u_1 + \cdots + \gamma_r u_r + \delta_1 w_1 + \cdots + \delta_t v_t \end{aligned}$$

for some $\alpha_i, \beta_j, \gamma_k, \delta_l \in F$. Thus $a + b$ is in the span of S.

On the other hand, suppose we have

$$\alpha_1 u_1 + \cdots \alpha_r u_r + \beta_1 v_1 + \cdots + \beta_s v_s + \gamma_1 w_1 + \cdots + \gamma_t w_t = 0$$

for some $\alpha_i, \beta_j, \gamma_k \in F$. Then the equation

$$\alpha_1 u_1 + \cdots \alpha_r u_r + \beta_1 v_1 + \cdots + \beta_s v_s = -(\gamma_1 w_1 + \cdots + \gamma_t w_t)$$

says that this vector lies in $W_1 \cap W_2$, which has $\{u_i\}$ as a basis. Since $\{u_i, v_j\}$ is linearly independent, we must have $\beta_j = 0$ for all $1 \leq j \leq s$. Since $\{u_i, w_k\}$ is linearly independent, we then have $\alpha_i = \gamma_k = 0$ for all $1 \leq i \leq r, 1 \leq k \leq t$ as well.

(2) Using the formula in (1), we have

$$\dim V = \dim W_1 + \dim W_2 = \dim(W_1 + W_2) - \dim(W_1 \cap W_2).$$

Thus

$$\begin{aligned} V = W_1 + W_2 &\iff \dim V = \dim(W_1 + W_2) \\ &\iff \dim(W_1 \cap W_2) = 0 \\ &\iff W_1 \cap W_2 = 0. \end{aligned}$$

(V) First we have $T(0_V) = 0_W \in U$. Thus $0_V \in T^{-1}(U)$.

Secondly suppose that $v_1, v_2 \in T^{-1}(U)$, i.e., $T(v_1), T(v_2) \in U$. (Remember the "function" T^{-1} does not exist in general.) Then $T(v_1 + v_2) = T(v_1) + T(v_2) \in U$. Thus $v_1 + v_2 \in T^{-1}(U)$.

Finally suppose that $v \in T^{-1}(U)$ and let $\alpha \in F$. Then $T(\alpha v) = \alpha T(v) \in U$. Therefore $\alpha v \in T^{-1}(U)$ as well.

(VI) (1) ($\Rightarrow$) Let $\{v_1, \cdots, v_m\}$ be a linearly independent set of V and suppose that

$$\alpha_1 T(v_1) + \cdots + \alpha_m T(v_m) = 0$$

for some $\alpha_i \in F$. Then $T(\alpha_1 v_1 + \cdots + \alpha_m v_m) = 0$, i.e., $\alpha_1 v_1 + \cdots + \alpha_m v_m \in \ker(T)$. Since T is 1-1, we have $\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$. Since $v_1, \cdots, v_m$ are linearly independent, we much have $\alpha_1 = \cdots = \alpha_m = 0$.

($\Leftarrow$) Suppose $T(v) = 0_W$ for some $v \in V$. Since 0_W is not linearly independent in W, v cannot be linearly independent in V by assumption. Thus $v = 0$ and T is 1-1.

(2) Let S be a generating set of V and let w be a vector in W.

($\Rightarrow$) Since T is onto, there exists $v \in V$ such that $T(v) = w$. Since S is a generating set, we have

$$v = \alpha_1 v_1 + \cdots + \alpha_r v_r$$

for some $\alpha_i \in F, v_i \in S$. Thus $T(v)$ is a linear combination of $\{T(v_i)\}_{1 \le i \le r}$ and hence $T(S)$ is a generating set of W.

($\Leftarrow$) Here we want to find $v \in V$ such that $T(v) = w$. Since $T(S)$ is a generating set of W by assumption, there exist $\alpha_1, \cdots, \alpha_n \in F$ and $v_1, \cdots, v_n \in S$ such that

$$w = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n).$$

Let $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$. Then $T(v) = w$ and we are done.

(3) ($\Rightarrow$) Since T is 1-1 and onto, T sends a basis to a basis by (1) and (2).

($\Leftarrow$) [This direction is not that trivial !] If a subset S of V is linearly independent, we can extend it to a basis S' ($S \subset S'$). By assumption $T(S')$ is a basis of W and in particular $T(S)$ is linearly independent. Thus T is 1-1 by (1).

On the other hand, if a subset S of V is a generating set, there exists a subset S' of S which forms a basis of V. Then $T(S')$ is a basis of W and in particular $T(S)$ is a generating set. Therefore T is onto by (2).

(VII) (1) Suppose $v_1 + W_1 = v_2 + W_1$. Then $v_1 - v_2 \in W_1$. Hence $T(v_1) - T(v_2) = T(v_1 - v_2) \in W_2$ by assumption. Therefore $T(v_1) + W_2 = T(v_2) + W_2$, which is what we want.

(2) We have $\overline{T}(0 + W_1) = T(0) + W_2 = 0 + W_2$.
Now if $v_1 + W_1, v_2 + W_2 \in V_1/W_1$, then we have

$$\begin{aligned} \overline{T}\big((v_1 + W_1) + (v_2 + W_1)\big) &= \overline{T}\big((v_1 + v_2) + W_1\big) \\ &= T(v_1 + v_2) + W_2 \\ &= \big(T(v_1) + T(v_2)\big) + W_2 \\ &= \big(T(v_1) + W_2\big) + \big(T(v_2) + W_2\big) \\ &= \overline{T}(v_1 + W_1) + \overline{T}(v_2 + W_1). \end{aligned}$$

Similarly if $v + W_1 \in V_1/W_1$ and $\alpha \in F$, then

$$\begin{aligned} \overline{T}\big(\alpha(v + W_1)\big) &= \overline{T}(\alpha v + W_1) \\ &= T(\alpha v) + W_2 \\ &= \alpha T(v) + W_2 \\ &= \alpha\big(T(v) + W_2\big) \\ &= \alpha\overline{T}(v + W_1). \end{aligned}$$

Thus $\overline{T}$ is linear.

(3) ($\Rightarrow$) Let $v \in T^{-1}(W_2)$, i.e., $T(v) \in W_2$. We want to show that $v \in W_1$.
We have $\overline{T}(v + W_1) = T(v) + W_2 = W_2$, i.e., $v + W_1 \in \ker(\overline{T})$. Since $\overline{T}$ is 1-1, we have $v + W_1 = W_1$ and hence $v \in W_1$.
($\Leftarrow$) Suppose that $\overline{T}(v+W_1) = 0$ for some $v+W_1 \in V_1/W_1$. Then $T(v)+W_2 = W_2$ and hence $T(v) \in W_2$. This means $v \in T^{-1}(W_2)$, which is a subset of W_1 by assumption. Thus $v + W_1 = W_1$ and $\overline{T}$ is 1-1.

(4) ($\Rightarrow$) Let $w \in V_2$. Since $\overline{T}$ is onto, there exists $v \in V_1$ such that

$$T(v) + W_2 = \overline{T}(v + W_1) = w + W_2.$$

Thus $w \in T(V_1) + W_2$, which is what we want.
($\Leftarrow$) Take any $w + W_2 \in V_2/W_2$. Since $V_2 = T(V_1) + W_2$, we can write

$$w = T(v) + w' \quad \text{for some } v \in V_1, w' \in W_2.$$

Then

$$\begin{aligned} \overline{T}(v + W_1) &= T(v) + W_2 \\ &= \big(T(v) + w'\big) + W_2 \quad (\text{since } w' \in W_2) \\ &= w + W_2. \end{aligned}$$

Thus $\overline{T}$ is onto.