There are 7 problems (I) - (VII) in total; some problems contain sub-problems, indexed by (1), (2), etc.

(I) [20%] True or false. (No explanation is needed.)

(1) Let F be a field. For any two positive integers m, n, the set of all $m \times n$ matrices is a vector space over F of dimension $m + n$.

(2) For any two finite dimensional vector spaces V and W over F, there exists a linear map T from V to W.

(3) If the linear map $T : V \to W$ is onto, then $\dim V \geq \dim W$.

(4) Let $T : V \to W$ be a linear map. Suppose that $\ker(T) = \{0\}$ and $T(v_1) = T(v_2)$. Then $v_1 = v_2$.

(5) There exists a linear map from $\mathbb{R}^2 \to \mathbb{R}^3$ sending $(1, 2)$ to $(2, 4, 6)$ and $(3, 6)$ to $(1, 3, 5)$.

(6) Let $T : V \to W$ be a linear map and S be a subset of V. If $T(S)$ is a subspace of W, then S is a subspace of V.

(7) Let S be a subset of a vector space V. Then the span of S is a subspace of V.

(8) If $\{u, v, w\}$ be a basis of V, then $\{u, u + v, u + v + w\}$ is also a basis.

(II) [15%]

(1) Give an example (and explanation) of a vector space V and two subspaces W_1 and W_2 of V such that the union $W_1 \cup W_2$ is not a subspace of V. You should not use the statement in the following question but give explicit indications.

(2) Let V be a vector space and W_1 and W_2 be subspaces of V. Show that if $W_1 \cup W_2$ is a subspace of V, then $W_1 \supset W_2$ or $W_1 \subset W_2$.

(III) [10%] Show that $\cos x$ and $\sin x$ are linearly independent (over $\mathbb{R}$) in the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(IV) [15%] Let V be a finite dimensional linear space and W_1, W_2 be two subspaces of V.

(1) Show that
$$\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2).$$

(2) Suppose that $\dim V = \dim W_1 + \dim W_2$. Show that $V = W_1 + W_2$ if and only if $W_1 \cap W_2 = \{0\}$.

(V) [10%] Let $T : V \to W$ be a linear map. Let U be a subspace of W. Show that the set $T^{-1}(U) = \{v \in V \mid T(v) \in U\}$ is a subspace of V.

(VI) [25%] Let $T : V \to W$ be a linear map.

(1) Show that T is 1-1 if and only if T sends any linearly independent set in V to a linearly independent set in W.

(2) Show that T is onto if and only if T sends any generating set of V to a generating set of W.

(3) Show that T is an isomorphism if and only if T sends any basis of V to a basis of W.

(VII) [35%] Recall that if W is a subspace of V over a field F, the *quotient* V/W is the set consisting of the cosets $v + W$ for all $v \in V$. Under the operations

$$\begin{aligned} (v_1 + W) + (v_2 + W) &= (v_1 + v_2) + W \quad (v_1, v_2 \in V) \\ \alpha(v + W) &= \alpha v + W \quad (\alpha \in F, v \in V), \end{aligned}$$

the quotient V/W is a vector space over F.

Now let $T : V_1 \to V_2$ be a linear map. Let $W_1 \subset V_1$ and $W_2 \subset V_2$ be subspaces. Suppose that $T(W_1) \subset W_2$. Define a map

$$\overline{T} : V_1/W_1 \to V_2/W_2$$

by requiring

$$\overline{T}(v + W_1) = T(v) + W_2.$$

(1) Show that $\overline{T}$ is well-defined, i.e. show that if $v_1 + W_1 = v_2 + W_1$, then $\overline{T}(v_1 + W_1) = \overline{T}(v_2 + W_1)$.

(2) Show that $\overline{T}$ is linear.

(3) Prove that $\overline{T}$ is 1-1 if and only if $W_1 \supset T^{-1}(W_2)$.

(4) Prove that $\overline{T}$ is onto if and only if $W_2 + T(V_1) = V_2$.