January 10, 2013

1. [15%] The characteristic polynomial $P(x)$ of A is

$$P(x) = \det \begin{pmatrix} x & 0 & 2 \\ -2 & x-1 & -4 \\ -1 & 0 & x-3 \end{pmatrix} = (x-1)^2(x-2).$$

The eigenspace E_1 consists of $v \in \mathbb{C}^3$ such that

$$0 = (I - A)v = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \\ -1 & 0 & -2 \end{pmatrix} v.$$

Thus E_1 is generated by the linearly independent $\{c_1, c_2\}$ where

$$c_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

On the other hand, E_2 is the set of solutions of

$$0 = (2I - A)v = \begin{pmatrix} 2 & 0 & 2 \\ -2 & 1 & -4 \\ -1 & 0 & -1 \end{pmatrix} v,$$

which is generated by

$$c_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore one can take

$$Q = (c_1, c_2, c_3) = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

2. (a) [5%] By the assumption, there exists an invertible $Q \in M_n(F)$ such that

$$A = Q^{-1}(\lambda I_n)Q = \lambda I_n.$$

   (b) [5%] The characteristic polynomial of A is $(x-\lambda)^n$. Thus by part (a), if A is invertible, $A = \lambda I_n$, which is a contradiction if $n > 1$.

3. (a) [5%] Notice that for any $A, B \in M_n(F)$, we have $(AB)^t = B^t A^t$. If A is invertible, we have

$$I = I^t = (AA^{-1})^t = (A^{-1})^t \cdot A^t.$$

   Thus A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

   (b) [5%] Taking determinants, we have

$$1 = \det(I) = \det(AA^t) = \det(A)\det(A^t) = \det(A)^2.$$

   Thus $\det(A) = \pm 1$.

4. (a) [5%] Let $v \in E_\lambda$. Since E_λ is a subspace, $T(v) = \lambda v \in E_\lambda$.

(b) [5%] Let $v \in W_1 \cap \cdots \cap W_r$. Since $v \in W_i$ for all $1 \le i \le r$, we have $T(v) \in W_i$. Thus $T(v) \in W_1 \cap \cdots \cap W_r$.

(c) [5%] Let $v \in W_1 + \cdots + W_r$. Then there exist $w_i \in W_i$ such that $v = w_1 + \cdots + w_r$. Since $T(w_i) \in W_i$, one has $T(v) = T(\sum w_i) = \sum T(w_i) \in \sum W_i$.

(d) [10%] Suppose W is T-invariant. Let $f \in W^\circ$ and $w \in W$. Then $T^*(f)(v) = f(T(v)) \in f(W) \subset W$. Thus W° is T^*-invariant.

Conversely suppose W° is T^*-invariant. Let $w \in W$ and $f \in W^\circ$. Consider the canonical isomorphism $\psi : V \to V^{\circ\circ}$. Then $\psi(T(w))(f) = f(T(w)) = T^*(f)(w) = 0$. Thus $\psi(T(W)) \subset W^{\circ\circ} = \psi(W)$ and hence $T(W) \subset W$.

(e) [5%] Suppose W is T-invariant. Since T is invertible, we must have $T(W) = W$. Thus $T^{-1}(W) = W$ and in particular W is T^{-1}-invariant.

By replacing T by T^{-1} in the above argument, we obtain the other direction.

5. (a) [10%] ($\Rightarrow$) If one takes a basis β_i for each W_i, then the disjoint union of β_i is a basis of V. Thus by counting elements, we obtain $\sum \dim W_i = \dim V$.

($\Leftarrow$) Again take a basis β_i for each W_i. Then the union β of these β_i is a generating set of $W_1 + \cdots + W_r = V$. However the number of elements in β is at most equal to the sum N of numbers of elements in β_i. By assumption $N = \sum \dim W_i = \dim V$. Thus β must be a basis and consequently $V = \bigoplus W_i$.

(b) [5%] Let V be the plane, $W_1 = x$-axis, $W_2 = y$-axis and $U =$ the diagonal line $x - y = 0$. Then $V = W_1 + W_2$. However $U \cap W_1 = U \cap W_2 = \{0\}$. Thus $U \neq (U \cap W_1) + (U \cap W_2)$.

6. (a) [5%] Suppose $f \in (W_1 + W_2)^\circ, w_i \in W_i$. Then $f(w_i) = 0$ since $w_i \in W_i \subset W_1 + W_2$. Thus $f \in W_1^\circ \cap W_2^\circ$.

Conversely suppose $g \in W_1^\circ \cap W_2^\circ$ and $v \in W_1 + W_2$. Then there exist $w_i \in W_i$ such that $v = w_1 + w_2$. One has $g(v) = g(w_1 + w_2) = g(w_1) + g(w_2) = 0$. Thus $g \in (W_1 + W_2)^\circ$.

(b) [10%] **Method I.** Let $f \in W_1^\circ + W_2^\circ$ and $v \in W_1 \cap W_2$. Then there exist $f_i \in W_i^\circ$ such that $f = f_1 + f_2$. Since $v \in W_1 \cap W_2 \subset W_i$, one has $f(v) = f_1(v) + f_2(v) = 0$. Thus we obtain $W_1^\circ + W_2^\circ \subset (W_1 \cap W_2)^\circ$.

Now we compare the dimensions. Let $n = \dim V$. One has

$$\begin{aligned}
\dim(W_1^\circ + W_2^\circ) &= \dim W_1^\circ + \dim W_2^\circ - \dim(W_1^\circ \cap W_2^\circ) \\
&= \dim W_1^\circ + \dim W_2^\circ - \dim(W_1 + W_2)^\circ \quad \text{(part (a))} \\
&= (n - \dim W_1) + (n - \dim W_2) - (n - \dim(W_1 + W_2)) \\
&= n - (\dim W_1 + \dim W_2 - \dim(W_1 + W_2)) \\
&= n - \dim(W_1 \cap W_2) \\
&= \dim(W_1 \cap W_2)^\circ.
\end{aligned}$$

Method II. Take a basis α of $W_1 \cap W_2$, and extend it to bases $\alpha \cup \beta_i$ of W_i. Then the disjoint union $\alpha \cup \beta_1 \cup \beta_2$ is a basis of $W_1 + W_2$. Extend it further to a basis $\beta = \alpha \cup \beta_1 \cup \beta_2 \cup \gamma$ of V and consider the dual basis $\beta^* = \alpha^* \cup \beta_1^* \cup \beta_2^* \cup \gamma^*$. Then by direct checking, $\beta_2^* \cup \gamma^*$ is a basis of W_1°, $\beta_1^* \cup \gamma^*$ a basis of W_2° and $\beta_1^* \cup \beta_2^* \cup \gamma^*$ a basis of $(W_1 \cap W_2)^\circ$. Thus $W_1^\circ + W_2^\circ = (W_1 \cap W_2)^\circ$.

Method III. Since two subspaces are the same if and only if their annihilators in the dual space are the same, we can just take the annihilators of $(W_1 \cap W_2)^\circ$ and $W_1^\circ + W_2^\circ$ in the double dual $V^{**} = V$ and apply part (a). Thus

$$(W_1^\circ + W_2^\circ)^\circ \overset{\text{part (a)}}{=} W_1^{\circ\circ} \cap W_2^{\circ\circ} = W_1 \cap W_2 = (W_1 \cap W_2)^{\circ\circ}.$$

(Here we use the canonical isomorphism $\psi : V \to V^{**}$ to identity V and V^{**}.)

7. (a) [5%] Let $v \in E_\lambda$. Then

$$T(S(v)) = S(T(v)) = S(\lambda v) = \lambda S(v).$$

Thus $S(v) \in E_\lambda$, which is what we need.

(b) [10%] This can be proved by induction. Suppose $\dim V = 1$. Then $S = \lambda I_1$ and $T = \mu I_1$ for some $\lambda, \mu \in F$.

Fix an integer $n > 1$. Assume that for any diagonalizable $S', T' \in \mathcal{L}(V')$ with $\dim V' < n$ and $S'T' = T'S'$, there exists an ordered basis β' such that $[S']_{\beta'}, [T']_{\beta'}$ are diagonal. Suppose we are given $S, T \in \mathcal{L}(V)$ with $\dim V = n$ and $ST = TS$.

Case 1. Suppose S, T both have only one eigenvalue. Then $S = \lambda I_n$ and $T = \mu I_n$ and hence their matrix representations are diagonal for any ordered basis of V.

Case 2. Suppose one of S and T, say T, has more then one eigenvalue. Let λ be an eigenvalue and V_1 be the associated eigenspace of T. Then

- $V = V_1 \oplus V_2$ where V_2 is the direct sum of other eigenspaces of T. In particular V_1 and V_2 are T-invariant.
- $1 \leq \dim V_1, \dim V_2 < n$ since T has more than one eigenvalue.
- V_1 and V_2 are also S-invariant by part (a).

Thus by induction applying to the restrictions of S, T to V_1 and V_2, there exist ordered bases β_1 and β_2 of V_1 and V_2, respectively, such that the matrix representations of the restrictions associated to β_i are diagonal. Since V is a direct sum of V_1 and V_2, the disjoint union $\beta = \beta_1 \cup \beta_2$ is an ordered basis of V. Then $[S]_\beta, [T]_\beta$ are diagonal.

8. (a) [10%] (i) For $\bar{T}$, we first show that the eigenspaces E_λ of T maps to eigenspaces of $\bar{T}$ or to $\{0\}$. Let $v \in E_\lambda$. Then $\bar{T}(\bar{v}) = \overline{T(v)} = \overline{\lambda v} = \lambda \bar{v}$. Thus $\bar{v}$ is in the space $\bar{E}_\lambda := \{u \in \bar{V} \mid \bar{T}(u) = \lambda u\}$, which is an eigenspace if it is non-zero.

Now since V is the sum of eigenspaces of T, the above argument shows that $\bar{V}$ is also the sum of eigenspaces of $\bar{T}$. Thus $\bar{T}$ is diagonalizable.

(ii) For T_W, Let $V = \bigoplus_{i=1}^r E_{\lambda_i}$ be the eigenspace decomposition of V. We show that $W = \bigoplus_{i=1}^r (W \cap E_{\lambda_i})$.

Let $w \in W$. Then there exist $v_i \in E_{\lambda_i}$ such that $w = \sum_{i=1}^r v_i$. Going to $\bar{V}$, we have $\sum_{i=1}^r \bar{v}_i = \bar{w} = 0$. However we already know that $\bar{V} = \bigoplus \bar{E}_{\lambda_i}$ from (i). Thus each $\bar{v}_i$ is zero in $\bar{V}$ and hence each v_i is in W.

(b) [10%] Take an ordered basis $\alpha = \{w_1, \cdots, w_m\}$ of W such that $[T_W]_\alpha$ is diagonal. Take $\gamma = \{v_1, \cdots, v_r\} \subset V$ such that $\bar{\gamma} := \{\bar{v}_1, \cdots, \bar{v}_r\}$ forms an ordered basis and $[\bar{T}]_{\bar{\gamma}}$ is diagonal. Then the disjoint union $\beta = \alpha \cup \gamma$ is an ordered basis of V and the matrix representation looks like

$$[T]_\beta = \begin{pmatrix} [T_W]_\alpha & * \\ 0 & [\bar{T}]_{\bar{\gamma}} \end{pmatrix}.$$

In the following, we replace γ by $\gamma' = \{v_1', \cdots, v_r'\}$ such that $\bar{v}_i' = \bar{v}_i$ in $\bar{V}$ and each v_i' is an eigenvector of T. Suppose for a fixed j that $T(v_j) = \sum c_i w_i + \mu v_j$. Write $T(w_i) = \lambda_i w_i$. What we need is to find $v_j' = v_j + \sum \epsilon_i w_i$ for some $\epsilon_i \in F$ such that $T(v_j') = \mu v_j'$, i.e. we want

$$\sum c_i w_i + \mu v_j + \sum \epsilon_i \lambda_i w_i = \mu v_j + \sum \mu \epsilon_i w_i.$$

By assumption $\mu \neq \lambda_i$ for all i. Thus if we set $\epsilon_i = \frac{c_i}{\mu - \lambda_i}$, then $\bar{v}_j'$ satisfies the requirement.

Now since $\bar{\gamma}' = \bar{\gamma}$, the disjoint union $\beta' = \alpha \cup \gamma'$ is a basis of V and consists of eigenvalues of T. Thus $[T]_{\beta'}$ is diagonal.