There are eight problems 1 $\sim$ 8 in total; some problems contain sub-problems, indexed by (a), (b), etc.

In the following, for a vector space V over a field F, denote $V^* = \mathcal{L}(V, F)$ the dual space of V. For $T \in \mathcal{L}(V, W)$, denote

$$\begin{array}{rcl} T^* : W^* & \to & V^* \\ f & \mapsto & f \circ T \end{array}$$

the transport of T. For a subset S of V, set

$$S^\circ = \{f \in V^* \mid f(s) = 0 \text{ for all } s \in S\}$$

to be the annihilator of S.

1. [15%] Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{pmatrix} \in M_3(\mathbb{C}).$$

   Find an invertible Q and a diagonal D in $M_3(\mathbb{C})$ such that $Q^{-1}AQ = D$.

2. [10%] Let $A \in M_n(F)$.

   (a) Suppose that A is diagonalizable and has only one eigenvalue λ. Show that $A = \lambda I_n$.

   (b) Show that the matrix

$$A = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}_{n \times n}$$

   (having λ in the diagonal, 1 above λ, and zero otherwise) is not diagonalizable for all $n > 1$.

3. [10%] Let $A \in M_n(F)$. Denote A^t the transport of A.

   (a) Show that if A is invertible, then A^t is also invertible and $(A^t)^{-1} = (A^{-1})^t$.

   (b) Show that if $AA^t = I_n$, then $\det(A) = \pm 1$.

4. [30%] Let $T \in \mathcal{L}(V)$. Recall that a subspace W of V is called T-invariant if $T(W) \subset W$.

   (a) Let $E_\lambda = \{v \in V \mid T(v) = \lambda v\}$ be the eigenspace of T associated with the eigenvalue λ. Show that E_λ is T-invariant.

   (b) Let $W_1, W_2, \cdots, W_r$ be T-invariant subspaces of V. Show that the intersection $W_1 \cap W_2 \cap \cdots \cap W_r$ is T-invariant.

   (c) Let $W_1, W_2, \cdots, W_r$ be T-invariant subspaces of V. Show that the sum $W_1 + W_2 + \cdots + W_r$ is T-invariant.

   (d) Show that a subspace W of V is T-invariant if and only if the annihilator W° is T^*-invariant

   (e) Suppose that T is invertible. Show that a subspace W of V is T-invariant if and only if W is T^{-1}-invariant.

5. [15%]

   (a) Let $W_1, W_2, \cdots, W_r$ be subspaces of V. Suppose $V = W_1 + W_2 + \cdots + W_r$. Show that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$ if and only if $\dim V = \sum_{i=1}^{r} \dim W_i$.

   (b) Give an example of a vector space V and three subspaces U, W_1, W_2 such that $V = W_1 + W_2$ but $U \neq (W_1 \cap U) + (W_2 \cap U)$. Remember to verify your answer (briefly).

6. [15%] Let W_1, W_2 be subspaces of V.

   (a) Show that $(W_1 + W_2)^\circ = W_1^\circ \cap W_2^\circ$.

   (b) Show that $(W_1 \cap W_2)^\circ = W_1^\circ + W_2^\circ$.

7. [15%] Let $S, T \in \mathcal{L}(V)$ satisfying $ST = TS$.

   (a) Let E_λ be the eigenspace of T associated with the eigenvalue λ. Show that $S(E_\lambda) \subset E_\lambda$.

   (b) Suppose that S, T are diagonalizable. Show that there exists an ordered basis β such that $[S]_\beta$ and $[T]_\beta$ are diagonal.

8. [20%] Let $T \in \mathcal{L}(V)$ and W a T-invariant subspace of V. Let $\bar{V} = V/W$ be the quotient of V by W. Denote $T_W \in \mathcal{L}(W)$ and $\bar{T} \in \mathcal{L}(\bar{V})$ the induced maps so that we have the commutative diagram

$$\begin{tikzcd} W \arrow[r, "\text{inclusion}"] \arrow[d, "T_W"'] & V \arrow[r, "\text{quotient}"] \arrow[d, "T"] & \bar{V} \arrow[d, "\bar{T}"] \\ W \arrow[r, "\text{inclusion}"] & V \arrow[r, "\text{quotient}"] & \bar{V}. \end{tikzcd}$$

   (a) Show that if T is diagonalizable, then T_W and $\bar{T}$ are diagonalizable.

   (b) Suppose that T_W and $\bar{T}$ do not have a common eigenvalue and both T_W and $\bar{T}$ are diagonalizable. Show that T is diagonalizable.