Homework Assignment 3

Due on May 5, 2011

Part I: [2, Chapter 9], Exercises 2 and Problems 2, 3, 4.

Part II:

1. (Cf. [3]) Let $t \in \mathbb{C}$ with $0 < |t| < 1$. Let

$$\phi(w) = \sum_{m=-\infty}^{\infty} \frac{t^m w}{(1 - t^m w)^2} - 2 \sum_{m=1}^{\infty} \frac{t^m}{(1 - t^m)^2}.$$

   (a) Show that the series $\phi(w)$ converges uniformly on each compact set. Thus $\phi(w)$ defines a meromorphic function on w with double poles at $w = t^m, m \in \mathbb{Z}$.

   (b) Now let $\tau \in \mathbb{H}$ and $t = e^{2\pi i \tau}$. Let $\wp(z)$ be the Weierstrass $\wp$-function associated with the lattice generated by $\{1, \tau\}$. Show that

$$\wp(z) = -4\pi \left(\phi(e^{2\pi i z}) + \frac{1}{12} \right).$$

   [Hint: Show that $\wp(z) + 4\pi\phi(e^{2\pi i z})$ is elliptic and has no poles.]

2. [1, p.275] Show that any elliptic function with periods ω_1, ω_2 can be written as

$$C \prod_{k=1}^{n} \frac{\sigma(z - a_k)}{\sigma(z - b_k)} \qquad (C = \text{const.}).$$

References

[1] L. V. Ahlfors, *Complex analysis. An introduction to the theory of analytic functions of one complex variable.* Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, 1978.

[2] E. Stein and R. Shakarchi, *Complex analysis.* Princeton Lectures in Analysis, II. Princeton University Press, Princeton, NJ, 2003.

[3] J. Tate, A review of non-Archimedean elliptic functions. *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*, 162-184, Ser. Number Theory, I, Int. Press, Cambridge, MA, 1995.