December 31, 2010

- [Exercise 5.17(b)] The formula should be

$$F(z) = \frac{b_0}{E'(\mathbf{0})} \frac{E(z)}{z} + \cdots ,$$

not

$$F(z) = \frac{b_0}{E'(\mathbf{z})} \frac{E(z)}{z} + \cdots .$$

- [Problem 5.1] We consider the case that $f(0) = 1$ and rearrange the zeros α_n of f such that $|\alpha_1| \leq |\alpha_2| \leq \cdots$. Let $\beta_n = |\alpha_n|$. Here we show that

$$\lim_{R \to 1^-} \sum_{n=1}^{\mathfrak{n}(R)} \log \frac{\beta_n}{R} = \sum_{n=1}^{\infty} \log \beta_n$$

where $\mathfrak{n}(R) = \max\{n | \beta < R\}$ for $0 \leq R < 1$. Indeed

$$\begin{aligned} \sum_{n=1}^{\mathfrak{n}(R)} \log \frac{\beta_n}{R} - \sum_{n=1}^{\mathfrak{n}(R)} \log \beta_n &= -\mathfrak{n}(R) \log R \\ &= \mathfrak{n}(R) \int_R^1 \frac{dr}{r} \\ &\leq \int_R^1 \mathfrak{n}(r) \frac{dr}{r}. \end{aligned}$$

The last term tends to 0 as $\int_0^1 \mathfrak{n}(r) \frac{dr}{r}$ is bounded by Jensen's formula and the boundedness of f.

- [Exercise 7.8] We show that the function

$$\xi(s) = \sqrt{\pi^{-s}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

is "of growth order $\rho = 1$ on the domain $\sigma = \Re(s) \geq \frac{1}{2}$".

(i) ($\rho \leq 1$) The first factor is harmless. For the second gamma factor, notice that, for $\sigma \geq 1/4$,

$$\int_1^{\infty} e^{-t} t^{\sigma - 1} dt \leq \max\{1, e^{\sigma \log(\sigma)}\} \quad \text{(cf. p.165)},$$

which implies that $\Gamma(s/2)$ is "of order ≤ 1 on $\sigma \geq 1/2$". For the third zeta factor, notice that on $\{\frac{1}{2} \leq \sigma \leq 2, |t| \geq 1\}$, we have $|\zeta(s)| \leq c|t|^2$ (cf. Prop.6.2.7), while on $\sigma \geq 2$, we have $|\zeta(s)| \leq \sum n^{-2}$.

(ii) ($\rho \geq 1$) We look at what happen on the positive real axis. The third factor $\zeta(\sigma)$ is ≥ 1 when $\sigma \gg 1$. On the other hand, at positive integers,

$$\begin{aligned} \log\left(\pi^{-\sigma} \Gamma(\sigma)|_{\sigma = n}\right) &= -n \log \pi + \log \Gamma(n) \\ &\sim n \log n \quad \text{(Ex.6.14)}. \end{aligned}$$

Thus $\rho \geq 1$.