Homework Assignment 2

Due on ?, 2011

In the following $\mathbb{F}_q$ denotes a finite field of q elements.

1. Let k be a field, $A = k[x_1, \cdots, x_n]$, and $\mathbb{A}^n$ the affine n-space. For any subset $S \subset A$, let $V(S) = \{\alpha \in \mathbb{A}^n \mid f(\alpha) = 0 \ \forall f \in S\}$.

   (a) Let I, J be two ideals of A. Show that
   $$V(I) \cup V(J) = V(IJ).$$

   (b) Let $\{I_j\}_{j \in J}$ be a collection of ideals of A. Show that
   $$\bigcap_{j \in J} V(I_j) = V\Big(\sum_{j \in J} I_j\Big).$$

2. Let $X = V(I) \subset \mathbb{A}^n$. Show that X is an irreducible topological space (under the Zariski topology) if and only if the radical $\sqrt{I}$ is a prime ideal.

3. Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$. Let $f(X, Y) = XY^p - YX^p - 1$ and $X = V(f) \subset \mathbb{A}^2$. Show that X is irreducible. [Hint: Use the change of variables $(X, Y) = (Z/T, 1/T)$ and then the Eisenstein's irreducibility criterion.]

4. Recall that any morphism $f : X \to Y$ between affine varieties over k induces naturally a k-algebra homomorphism $f^* : k[Y] \to k[X]$ between their coordinates rings. Now let $g : Y \to Z$ be another morphism of affine varieties over k. Show that
   $$(g \circ f)^* = f^* \circ g^*$$
   as k-algebra homomorphisms from $k[Z]$ to $k[X]$.

5. Let $X \subset \mathbb{A}^n$ be an algebraic set. Let $\overline{X} \subset \mathbb{P}^n$ be the Zariski closure of X in $\mathbb{P}^n$ (via the inclusions $X \subset \mathbb{A}^n \subset \mathbb{P}^n$). Show that X is irreducible in $\mathbb{A}^n$ if and only if $\overline{X}$ is irreducible in $\mathbb{P}^n$.