Assignment 3

Due on December 31, 2009

1. Let $K = \overline{\mathbb{Q}}_p$, $G = \mathrm{Gal}(\overline{K}/K)$ and E a Tate curve over K (with $E(K) = K^\times/t^{\mathbb{Z}}$ for some $t \in K, 0 < |t| < 1$). Fix a positive integer n; denote by E_n the p^n-torsion points of the group $E(\overline{K})$. Thus E_n is a free $\mathbb{Z}/p^n\mathbb{Z}$-module of rank 2 and is equipped with a natural G-action.

   (a) Show that there is an exact sequence of representations of G

   $$1 \to \mu_{p^n} \to E_n \to \mathbb{Z}/p^n\mathbb{Z} \to 0, \qquad (1)$$

   where μ_{p^n} = the p^n-th roots of 1 (with the natural $\mathbb{Z}/p^n\mathbb{Z}$- and G-action) and G acts on $\mathbb{Z}/p^n\mathbb{Z}$ trivially.[1]

   (b) The sequence (1) corresponds to a class in the group cohomology $H^1(G, \mu_{p^n})$ as follows. Regard E_n as an additive $\mathbb{Z}/p^n\mathbb{Z}$-module. Take $u, v \in E_n$ such that u and the image of v generate μ_{p^n} and $\mathbb{Z}/p^n\mathbb{Z}$, respectively. Consider the map $g \mapsto \alpha(g), g \in G$ defined by

   $$g(v) = v + \alpha(g)u.$$

   Show that c defines a class in $H^1(G, \mu_{p^n})$; more precisely, verify that c is a cocycle with values in μ_{p^n} and does not depend on the choice of u, v up to a coboundary.

2. Recall that for $f(z) = \alpha z^\nu(1 + a_1 z + a_2 z^2 + \cdots) \in \mathbb{C}_p[[z]], \alpha \neq 0$, we define the *norm* of f by

   $$\|f\| = \sup_n \{|a_n|^{1/n}\};$$

   $U_f := \{z \in \mathbb{C}_p \mid |z| < \|f\|^{-1}\}$ is the *inner circle of convergence* of f. Find the norm of the series

   $$f(z) = \log(1 + z).$$

   Conclude that the two functions $\log z$ and $\exp z$ establish an isomorphism between the multiplicative group $1 + U_f$ and the additive group U_f.

3. Let K be a complete non-archimedean valuation field. For $i = 1, 2$, let E_i/K be the Tate curves with periods $t_i \in K, 0 < |t_i| < 1$. Verify that $\mathrm{Hom}_K(E_1, E_2) = 0$ or $\cong \mathbb{Z}$.

[1] Taking projective limit over n, we obtain an exact sequence of p-adic representations of G :

$$1 \to \mathbb{Z}_p(1) \to T_p E \to \mathbb{Z}_p \to 1,$$

where $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$ and $T_p E$ is the (p-adic) Tate module of E.