Assignment 1

Due on October 22, 2009

1. Show that $\overline{\mathbb{Q}}_p$ is not complete (under the p-adic topology).
2. Find out the roots of $f(X) = \log(1+X)$ in the disc $D(1)$ (of radius 1) of $\mathbb{C}_p$.

Define the *Möbius function* $\mu : \mathbb{Z}_{>0} \to \{0, \pm 1\}$ by the following three rules:

- $\mu(1) = 1$.
- For a prime p and $k \in \mathbb{Z}_{>0}$,

$$\mu(p^k) = \begin{cases} -1 & \text{if } k = 1 \\ 0 & \text{if } k > 1. \end{cases}$$

- For $(m, n) = 1$, $\mu(mn) = \mu(m)\mu(n)$.

3. Show that
   (a) For all $n > 1$,
   $$\sum_{d|n} \mu(d) = 0.$$
   (b) Formally (i.e. in $\mathbb{Q}[[X]]$ in this case) we have
   $$\exp(X) = \prod_{n=1}^{\infty} (1 - X^n)^{-\frac{\mu(n)}{n}}.$$
   (c) Fix a prime p. Formally we have the equality for the Artin-Hasse exponential[1]
   $$E_p(X) := \exp\left(\sum_{k=0}^{\infty} \frac{X^{p^k}}{p^k}\right) = \prod_{(n,p)=1} (1 - X^n)^{-\frac{\mu(n)}{n}}.$$

   (Hint: For (b, c), take log.)

[1](For fun; no credit) From Wikipedia: "The coefficient of X^n of $n!E_p(X)$ is the number of elements of the symmetric group on n points of order a power of p."