Assignment 3

Due on December 24, 2009

1. ([1, Ex 6.43]) Let $\pi : E \to M$ be an oriented rank 2 bundle. As we saw in the proof of the Thom isomorphism, wedging with the Thom class is an isomorphism $\wedge\Phi : H^*(M) \to H^{*+2}_{cv}(E)$. Therefore every cohomology class on E is the wedge product of Φ with the pullback of a cohomology class on M. Find the class u on M such that
$$\Phi^2 = \Phi \wedge u \quad \text{in } H^*_{cv}(E).$$

2. ([1, Ex 6.45]) On the complex projective space $\mathbb{C}P^n$ there is a tautological line bundle S, called the *universal subbundle*; it is the subbundle of the product bundle $\mathbb{C}P^n \times \mathbb{C}^{n+1}$ given by
$$S = \{(\ell, z) \mid z \in \ell\}.$$
Above each point ℓ in $\mathbb{C}P^n$, the fiber of S is the line represented by ℓ. Find the transition functions of the universal subbundle S of $\mathbb{C}P^1$ relative to the standard open cover and compute its Euler class.

3. ([1, Ex 6.46]) Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and i the antipodal map on S^n:
$$i : (x_1, \dots, x_{n+1}) \to (-x_1, \dots, -x_{n+1}).$$
The *real projective space* $\mathbb{R}P^n$ is the quotient of S^n by the equivalent relation[1]
$$x \sim i(x), \quad \text{for } x \in \mathbb{R}^{n+1}.$$
   (a) An *invariant form* on S^n is a form ω such that $i^*\omega = \omega$. The vector space of invariant forms on S^n, denoted $\Omega^*(S^n)^I$, is s differential complex, and so the invariant cohomology $H^*(S^n)^I$ of S^n is defined. Show that $H^*(\mathbb{R}P^n) \simeq H^*(S^n)^I$.
   (b) Show that the natural map $H^*(S^n)^I \to H^*(S^n)$ is injective. [*Hint :* If ω is an invariant form and $\omega = d\tau$ for some form τ on S^n, then $\omega = d(\tau + i^*\tau)/2$.]
   (c) Give S^n its standard orientation (p. 70). Show that the antipodal map $i : S^n \to S^n$ is orientation-preserving for n odd and orientation-reserving for n even. Hence, if $[\sigma]$ is a generator of $H^n(S^n)$, then $[\sigma]$ is a nontrivial invariant cohomology class if and only if n is odd.
   (d) Show that the de Rham cohomology of $\mathbb{R}P^n$ is
$$H^q(\mathbb{R}P^n) = \begin{cases} \mathbb{R} & \text{for } q = 0, \\ 0 & \text{for } 0 < q < n, \\ \mathbb{R} & \text{for } q = n \text{ odd}, \\ 0 & \text{for } q = n \text{ even}. \end{cases}$$

4. ([1, Ex 11.19]) Show that the Euler class of an oriented sphere bundle with even-dimensional fibers is zero, at least when the sphere bundle comes from a vector bundle.

[1] Convince yourself that $\mathbb{R}P^n$ is a manifold.

5. ([1, Ex 11.21]) Compute the Euler class of the unit tangent bundle of the sphere S^k by finding a vector field on S^k and computing its local degrees.

6. ([1, Ex 11.26]) (*Lefschetz fixed-point formula*). Let $f : M \to M$ be a smooth map of a compact oriented manifold into itself. Denote by $H^q(f)$ the induced map on the cohomology $H^q(M)$. The *Lefschetz number* of f is defined to be

$$L(f) = \sum_q (-1)^q \operatorname{trace} H^q(f).$$

   Let Γ be the graph of f in $M \times M$.

   (a) Show that

   $$\int_\Delta \eta_\Gamma = L(f).$$

   (b) Show that if f has no fixed points, then $L(f)$ is zero.

   (c) At a fixed point P of f the derivative $(Df)_P$ is an endomorphism of the tangent space $T_P M$. We define the *multiplicity* of the fixed point P to be

   $$\sigma_P = \operatorname{sgn} \det((Df)_P - I).$$

   Show that if the graph Γ is transversal to the diagonal Δ in $M \times M$, then

   $$L(f) = \sum_P \sigma_P,$$

   where P ranges over the fixed points of f. (For an explanation of the meaning of the multiplicity σ_P, see Guillemin and Pollack [2, p. 121].)

References

[1] R. Bott abd L.W. Tu, *Differential forms in algebraic topology.* Graduate Texts in Mathematics, 82. Springer-Verlag, New York-Berlin, 1982.

[2] Guillemin, V. and Pollack, A. *Differential Topology*, Prentice-Hall, Englewood Cliffs, New Jersey, 1974.