Assignment 1

Due on October 15, 2009

1. ([1, Ex 5.16]) Let x, y be the standard coordinates and r, θ the polar coordinates on $M = \mathbb{R}^2 \setminus \{0\}$.
   (a) Show that the Poincaré dual of the ray $\{(x, 0) \mid x > 0\}$ in M is $d\theta/2\pi$ in $H^1(M)$.
   (b) Show that the closed Poincaré dual of the unit circle in $H^1(M)$ is 0, but the compact Poincaré dual is the non-trivial generator $\rho(r)dr$ in $H^1_c(M)$ where ρ is a bump function with total integral 1. (By a bump function we mean a smooth function whose support is concetrated in some disc and whose graph looks like a "bump".)
2. Compute the de Rham cohomology groups $H^*(M)$ and $H^*_c(M)$ of $M = \mathbb{R}^2 \setminus \{0, 1\}$ by using (1) the Mayer-Vietoris sequence; (2) elementary calculus (to find out an explicit basis).
3. Compute the de Rham cohomology groups of the torus $T = S^1 \times S^1$ by using (1) the Mayer-Vietoris sequence; (2) the Künneth formula.

Notation: For a homomorphism $f : A \to B$ of abelian groups, let $\ker f$ = the kernel of f; $\operatorname{im} f$ = the image of f; $\operatorname{coker} f$ = the cokernel of $f = B/\operatorname{im} f$.

4. Prove the following *snake lemma* : Suppose we have a commutative diagram of abelian groups

$$\begin{array}{ccccccc} & & A & \xrightarrow{p} & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \xrightarrow{q} & C' & & \end{array}$$

   whose two rows are exact. Then (1) there exists a natural homomorphism

$$\delta : \ker h \to \operatorname{coker} f$$

   such that (2) the sequence

$$\begin{array}{ccccc} \ker f \longrightarrow \ker g \longrightarrow \ker h & & \\ \downarrow \delta & & \\ \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h & & \end{array}$$

   is exact. Furthermore, (3) if p is injective, then $\ker f \to \ker g$ is injective; (4) if q is surjective, the $\operatorname{coker} g \to \operatorname{coker} h$ is surjective.
5. ([1, Ex 5.5]) Prove the Five Lemma: given a commutative diagram of abelian groups and group homomorphisms

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E & \longrightarrow & \cdots \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \epsilon\downarrow & & \\ \cdots & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' & \longrightarrow & \cdots \end{array}$$

   in which the rows are exact, if the maps α, β, δ and ϵ are isomorphisms, then so is the middle one γ. (Hint: One can prove this either directly or by applying the snake lemma.)

Recall that a *complex* $(A_i, \delta_i)_{i\in\mathbb{Z}}$ of abelian groups is a chain of homomorphisms of abelian groups

$$\cdots \longrightarrow A_{i-1} \xrightarrow{\delta_{i-1}} A_i \xrightarrow{\delta_i} A_{i+1} \longrightarrow \cdots$$

such that $\delta_i \circ \delta_{i-1} = 0$ (i.e. the trivial map) for all i. Let $\underline{A} = (A_i, \delta_i)$ be a complex. The *i-th cohomology* $H^i(\underline{A})$ of A is define by

$$H^i(\underline{A}) = \ker \delta_i / \operatorname{im} \delta_{i-1}.$$

Let $\underline{B} = (B_i, \epsilon_i)$ be another complex. A *morphism* $\underline{f} : \underline{A} \to \underline{B}$ from $\underline{A}$ to $\underline{B}$ is a collection of homomorphisms $f_i : A_i \to B_i$ such that the diagrams

$$\begin{array}{ccc} A_i & \xrightarrow{\delta_i} & A_{i+1} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i+1}} \\ B_i & \xrightarrow{\epsilon_i} & B_{i+1} \end{array}$$

commute for all i. A chain of morphisms $\underline{A} \to \underline{B} \to \underline{C}$ of complexes is called *exact* if $A_i \to B_i \to C_i$ is exact for each i.

6. Consider a short exact sequence of complexes

$$0 \to \underline{A} \to \underline{B} \to \underline{C} \to 0.$$

Then (1) there exists a chain

$$\cdots \to H^i(\underline{A}) \to H^i(\underline{B}) \to H^i(\underline{C}) \to H^{i+1}(\underline{A}) \to H^{i+1}(\underline{B}) \to \cdots$$

of group homomorphisms such that (2) it is exact. (Hint: One can construct the maps and prove the exactness either directly or by applying the snake lemma.)

References

[1] R. Bott abd L.W. Tu, *Differential forms in algebraic topology.* Graduate Texts in Mathematics, 82. Springer-Verlag, New York-Berlin, 1982.