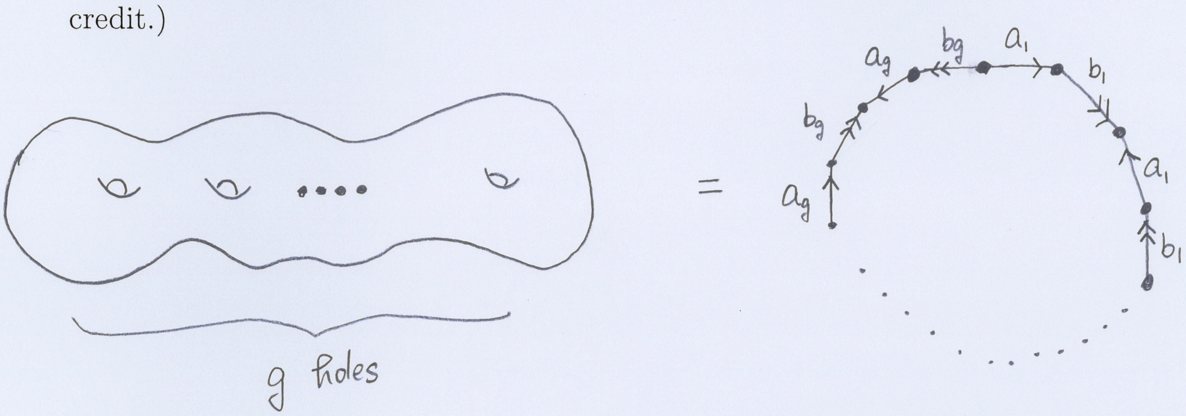
Midterm Exam

November 12, 2009

There are 5 problems below with points indicated; the maximal total points you can get is set to be 30.

1. (10pts) Compute the de Rham cohomology groups of the following compact, orientable surface with g holes in $\mathbb{R}^3$. (If this is too hard, do the case $g = 2$ for partial credit.)



2. (10pts) Let $p_1, \cdots, p_r$ be r distinct points on the torus $T = S^1 \times S^1$ ($r \geq 1$). Let $M = T \setminus \{p_1, \cdots, p_r\}$. Find the de Rham cohomology groups $H^*(M)$ and $H^*_c(M)$. (If this is too hard, do $r = 1$ for partial credit.)

3. (5pts) For a manifold M (with a finite good cover) of dimension n, let

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{R}} H^i(M) \quad ; \quad \chi_c(M) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{R}} H^i_c(M).$$

Does $\chi(M) = \chi_c(M)$ always hold true? If yes, prove it; if not, give a counterexample.

4. (5pts) Let X_1, X_2 be two manifolds and $Y_i \subset X_i, i = 1, 2$, be closed submanifolds. Let η_i be the Poincaré duals of Y_i in $H^*(X_i)$. What is the Poincaré dual of $Y_1 \times Y_2$ in $H^*(X_1 \times X_2) = H^*(X_1) \otimes H^*(X_2)$? Verify your answer.

We use the following convention: A double complex K (will be assumed to be concentrated in the first quadrant and) is equipped with two differentials $\delta : K^{p,q} \to K^{p+1,q}$ and $d : K^{p,q} \to K^{p,q+1}$. A homomorphism $f : K \to K'$ consists of maps $f : K^{p,q} \to (K')^{p,q}$ which commutate with δ and d.

5. (10pts) Prove the statement[1]: Whenever a homomorphism $f : K \to K'$ of double complexes induces H_d-isomorphism, it also induces H_D-isomorphism. (I.e., "$f : H(K^{p,\bullet}, d) \to H((K')^{p,\bullet}, d)$ *are isomorphisms for all* p" implies "$f : H(K, D) \to H(K', D)$ *is an isomorphism*" where $D = \delta + (-1)^p d$ is the differential on the associated single complex of K.)

[1]Bott and Tu, Exercise 9.13