Summer Seminar 2013

Étale Fundamental Groups

July 24, 2013

The aim of this seminar is to learn ... Of course, one should also look at the classic SGA 1 [2]. One very important topic that will not be covered is the theory of ℓ-adic étale sheaves and their cohomology. For this see, for example, [1].

The format of the seminar is as follows. We divide the theory into several topics. We shall have a one-day meeting every week discussing one topic. During the meeting day, there will be a discussion/introductory session where we explain the ideas, structures and/or the formulations in applications of the topic. After the discussion session, there will be two lectures (usually given by students) where we work out more details for selected properties, theorems, etc, of the topic.

The topics are listed below.

(*a*) *Schemes*

This is a warmup (in a hot day!) for the learning seminar.

Discussion: Explain the ideas of schemes (a ringed space which is locally a commutative ring).

Lectures: Zariski topology; open = localization; examples, including $\mathrm{Spec}(\mathbb{Z}), \mathbb{A}^n, \mathbb{P}^n$, hypersurfaces and maybe also toric varieties; tensor product over a scheme; explain "points = solutions of equations". The basic reference is of course [3, Chap. 2].

(*b*) *Galois categories*

One way to obtain the underlying symmetry of a geometric object X is to put everything we are interested over X into a 'category' $\mathcal{C}$. Then one studied the extra structures of $\mathcal{C}$ and the symmetry reveals itself. This is the approach we will use here.

Discussion: Examples from classical Galois theory and topology. Maybe also mention some modern topics like Galois theory for ordinary linear differential equations and the Tannaka duality where this categorical approach occurs too.

Lectures: Give details in [4, Chap. 3].

(*c*) *Morphisms*

Maybe we need two days for this?

Discussion: Explain the need to have étale covers in order to distinguish points like Spec(k) for a field k. Also this étale localization makes smooth closed points look the same.

Lectures: (i) Definitions and properties of smooth, flat, unramified and étale morphisms; their relations with lifting properties and with differentials. (ii) Prove that the collection of étale covers of a fixed scheme X forms a Galois category and hence one obtains the étale fundamental group.

(d) *Properties/Examples/Applications I*

First give some simple examples of π_1, e.g., for $\mathbb{Z}_p$ and a non-reduced scheme.

Discuss the homotopy exact sequences. Specifically, give proofs of Prop. 5.6.1, 5.6.4, 5.6.6 in [6].

Discuss the specialization map.

(e) *Properties/Examples/Applications II*

Prove the finite-generation property ([6, Prop.5.7.1, Thm.5.7.4]). Discuss the fundamental groups of curves ([6, Thm.5.7.13]) and abelian varieties ([6, Thm.5.6.10]). Discuss Lang's theorem [6, Thm.5.8.16].

References

[1] L. Fu, *Etale cohomology theory*. Nankai Tracts in Mathematics, 13. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. (E-book available through `NTU tulips`)

[2] A. Grothendieck, *Revêtements étales et groupe fondamental*. 1971. Various versions are available online.

[3] R. Hartshorne, *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[4] H. W. Lenstra, *Galois theory for schemes*. 1985. Available online.

[5] J. P. Murre, *Lectures on an introduction to Grothendieck's theory of the fundamental group*. 1967, Available online.

[6] T. Szamuely, *Galois groups and fundamental groups*. Cambridge Studies in Advanced Mathematics, 117. Cambridge University Press, Cambridge, 2009.