Variation of the Unit Root along Certain Families of Calabi-Yau Varieties*†

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Abstract

In this article, we explain a method of studying the variation of the zeta functions of Calabi-Yau varieties parametrized by a smooth affine space over a finite field. The method is based on the crystalline interpretation by Katz of Dwork's work. The relevant definitions of crystals over a smooth base are briefly reviewed. For such a crystal, we recall the relation between the unit-root sub-crystal and the horizontal sections of a lifted family in the Calabi-Yau situation. Two examples on families of Calabi-Yau varieties are given to illustrate the application of this relation. We also present some questions arising through the study of these examples.

1 Introduction

Let k be a finite field of characteristic p with $q = p^a$ elements. Let $\bar{k}$ be an algebraic closure of k. Let X be a smooth, geometrically irreducible hypersurface in $\mathbb{P}^{n+1}$ defined by a degree $(n+2)$ polynomial. Then X is a Calabi-Yau variety over k. For any positive integer r, let k_r be a field extension of k of degree r. Let N_r be the number of k_r-points $X(k_r)$ of X. Then the zeta function of X over k

$$Z(X/k,T) := \exp\left\{\sum_{r=1}^{\infty} N_r \frac{T^r}{r}\right\}$$

is a rational function in T of the form

$$Z(X/k,T) = \frac{[\det\left(1 - T\cdot F|H^n_{et}(X_{\bar{k}},\mathbb{Q}_\ell)\right)]^{(-1)^{n+1}}}{\prod_{i=0}^{n}(1-q^iT)}.$$

Here F is the geometric frobenius acting on the ℓ-adic étale cohomology $H^n_{et}(X_{\bar{k}},\mathbb{Q}_\ell)$ for some prime $\ell \neq p$ of $X_{\bar{k}} = X \times_k \bar{k}$. We have an equality of the characteristic polynomials over $\mathbb{Z}$

$$\det\big(T - F|H^n_{et}(X_{\bar{k}},\mathbb{Q}_\ell)\big) = \det\big(T - \phi^a|H^n_{cris}(X/W)\big), \tag{1}$$

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where W is the Witt ring of k and ϕ is the absolute frobenius on the crystalline cohomology $H^n_{cris}(X/W)$ of X. Under the degree condition on the defining polynomial of X, it is known that there is at most one root π of this polynomial which is a p-adic unit (with respect to a p-adic valuation on $\bar{\mathbb{Q}}$). If such a π exists, then π is an element in W via the right side of (1), and we call π *the unit root of* X.

In this article, we are interested in the variation of the unit root along a family of Calabi-Yau hypersurfaces. Suppose that X_t is a smooth and fiber-wise geometrically irreducible family of Calabi-Yau hypersurfaces over a smooth affine k-scheme S_0. Assume further that for each finite field extension k' of k and any k'-point λ_0 of S_0, the corresponding X_{λ_0} has the unit root π_{λ_0}. Then the fiber-wise unit roots form a sub-crystal U ([6], Theorem (2.4.2)) in the relative crystalline cohomology $H^n_{cris}(X_0/(S_0/W))$. This unit-root part U can be regarded as the collection of the information of π_{λ_0} for various points λ_0.

The interesting observation due to Dwork (see [5] for the crystalline interpretation) is that if there exists a nice lifting of the whole family X_t to characteristic zero, then this unit-root part U is generated locally by the horizontal section with respect to the Gauss-Manin connection ∇ associated to the lifted family. Moreover this local description can be used to compute the unit roots π_{λ_0} for *all* possible values of λ_0 in terms of a *fixed* local expression of the solution with respect to a basis. Thus the geometric origin of the Picard-Fuchs differential equation $\mathcal{L}$ gives rise to an analytic continuation property on the solutions to $\mathcal{L}$. On the other hand, if there is a unique solution to $\mathcal{L}$, the unit-root part U can be recovered totally in terms of the differential equation $\mathcal{L}$, and it then gives an analytic formula for π_{λ_0} provided that one finds an explicit expression of a ∇-horizontal section. (See §§3 and 4 below for examples.)

We explain the above observation in terms of the associated Hodge F-crystal of the lifted family (due to Katz, [5]). We review the notation of Hodge F-crystals and the interrelation between the variation of the unit root along a family and the ∇-horizontal sections for the lifted family to characteristic zero in §2. In the following two sections, we give concrete examples to illustrate the application of this relation. Besides the formulae for the unit roots of the fibers of the families, new observations are obtained while studying these examples. In §3, a family of elliptic curves is presented. This family is not of hypergeometric type but an analogous congruent property for the coefficients of certain hypergeometric series seems to hold in this case (see Conjecture 3.4). In §4, we give a new interpretation for one of the Calab-Yau conditions (see equation (13)) on an ordinary linear differential equation of order four. We then apply this observation to the famous family of the deformation of the Fermat quintic threefold. Finally we list some open questions concerning the structure of the local deformation of a generic member in the family.

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Notations. In this article, we let k be a perfect field of characteristic $p > 0$. Let $W = W(k)$ be the ring of Witt vectors of k. Denote by σ the absolute frobenius on k and W. Recall that the *hypergeometric series* ${}_mF_n$ with upper parameters $\{a_i\}_{i=1}^m$ and lower parameters $\{b_i\}_{i=1}^n$, $a_i, b_i \in \mathbb{C}$ and $b_i \notin \mathbb{Z}_{\leq 0}$, is defined by the formal power series

$$ {}_mF_n\left(\begin{array}{c} a_1, a_2, \cdots, a_m \\ b_1, b_2, \cdots, b_n \end{array} ; x \right) = \sum_{r=0}^{\infty} \frac{(a_1)_r (a_2)_r \cdots (a_m)_r}{(b_1)_r (b_2)_r \cdots (b_n)_r} \cdot \frac{x^r}{r!}, $$

where $(a)_0 = 1$ and $(a)_r = a(a+1)\cdots(a+r-1)$ for $r > 0$ is the *Pochhammer symbol.* Notice that if a_i is a non-positive integer for some i, the series ${}_mF_n$ is a polynomial. We may regard ${}_mF_n$ as a formal power series over a ring R whenever the expansion makes sense over R.

2 Crystals

We recall the definition of F-crystals over a smooth base. Roughly speaking, one thinks of an F-crystal as a cohomology group (the crystalline cohomology) $\mathcal{M}$ with an absolute frobenius structure of a scheme over a base of positive characteristic with nice properties. A Hodge F-crystal is then the identification of $\mathcal{M}$ with the relative de Rham cohomology of a nice lifting of the whole family to characteristic zero. It may be regarded as a bridge connecting the characteristic $p > 0$ and characteristic zero worlds. To illustrate this phenomenon, we describe the relation between the variation of the unit root along a family of Calabi-Yau varieties over a finite field k and the horizontal section of a lifted family over the ring of Witt vectors W of k with respect to the Gauss-Manin connection. The material in this section is mainly taken from [5].

(a) Hodge F-crystals over W

Definition. Let k be a perfect field, and W be the ring of Witt vectors of k. Denote by σ the absolute frobenius on k and W.

(i) An *F-crystal over W* is a free W-module of finite rank together with a σ-linear endomorphism

$$\phi : M \to M$$

such that ϕ induces an isomorphism on $M \otimes_{\mathbb{Z}} \mathbb{Q}$. It is called a *unit-root F-crystal* if ϕ is an isomorphism. The *rank* of the F-crystal M is the rank of M as a W-module.

(ii) A *Hodge F-crystal over W* is an F-crystal M over W together with a decreasing filtration $\mathrm{Fil}^\bullet M$ on M indexed by $\mathbb{Z}$ such that

(a) $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^n M = 0$ for $n >> 0$,

(b) $\mathrm{Fil}^i M$ and $\mathrm{Fil}^i M / \mathrm{Fil}^{i+1} M$ are free W-modules for all i, and

(c) (divisibility) $\phi(\mathrm{Fil}^i M) \subset p^i M$ for all i.

Example. The examples of interest to us come from cohomology groups of geometric objects. Here is a simple one.

(i) Let X_0 be a smooth hypersurface in $\mathbb{P}^{n+1}$ over k. Then the crystalline cohomology $M := H^n_{cris}(X_0/W)$ is an F-crystal over W.

(ii) Suppose that X_0 is the special fiber of a smooth hypersurface X in $\mathbb{P}^{n+1}$ over W. Then we have a canonical identification

$$H^n_{dR}(X/W) = H^n_{cris}(X_0/W).$$

The Hodge to de Rham spectral sequence on $H^n_{dR}(X/W)$ degenerates and it provides a filtration on M. With these data, M is then a Hodge F-crystal over W.

(b) Hodge F-crystals over a smooth base

Now we turn to the variation of F-crystals. For simplicity, we will restrict ourselves to the following setting which is enough for the subsequent discussion in this article.

Let $A = W[t][\mathcal{H}^{-1}]$ where $\mathcal{H} = \mathcal{H}(t) \in W[t]$ such that $\mathcal{H}$ is not divisible by p. Thus $\operatorname{Spec} A$ is the space obtained by removing some points from the affine line over W. Let $A_n = A/p^{n+1}A$ for all non-negative integers n and let $A_\infty = \varprojlim A_n$ be the projective limit. Let $\Omega^c_{A_\infty/W}$ be the space of continuous derivatives of A_∞ over W. We extend the absolute frobenius σ on W to A_∞ by setting $\sigma(t) = t^p$. We usually write $\sigma(a)$ as a^σ for $a \in A_\infty$.

Definition. Let A and $A_n, n = 0, 1, \ldots, \infty$, be as above.

(i) An *F-crystal over A_∞* is a locally free A_∞-module $\mathcal{M}$ of finite rank together with a (p-adically) topologically nilpotent integral connection

$$\nabla : \mathcal{M} \to \mathcal{M} \otimes_{A_\infty} \Omega^c_{A_\infty/W}$$

and a horizontal σ-linear endomorphism

$$\phi : \mathcal{M} \to \mathcal{M}$$

such that ϕ induces an isomorphism on $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Q}$. The *rank* of the F-crystal $\mathcal{M}$ is the rank of $\mathcal{M}$ as an A_∞-module.

(ii) A *Hodge F-crystal over A_∞* is an F-crystal $\mathcal{M}$ over A_∞ together with a filtration $\operatorname{Fil}^\bullet \mathcal{M}$ on $\mathcal{M}$ indexed by $\mathbb{Z}$ such that

(a) $\operatorname{Fil}^0 \mathcal{M} = \mathcal{M}$ and $\operatorname{Fil}^n \mathcal{M} = 0$ for $n >> 0$,

(b) $\operatorname{Fil}^i \mathcal{M}$ and $\operatorname{Fil}^i \mathcal{M}/\operatorname{Fil}^{i+1} \mathcal{M}$ are locally free A_∞-modules for all i,

(c) (divisibility) $\phi(\operatorname{Fil}^i \mathcal{M}) \subset p^i \mathcal{M}$ for all i, and

(d) (transversality) $\nabla(\operatorname{Fil}^i \mathcal{M}) \subset (\operatorname{Fil}^{i-1} \mathcal{M}) \otimes_{A_\infty} \Omega^c_{A_\infty/W}$ for all i.

Suppose $\mathcal{M}$ is an F-crystal over A_∞. Let k' be a perfect field extension of k and W' the Witt ring of k'. Suppose $e_0 : A_0 \to k'$ is a k-morphism with $e_0(t) = \lambda_0$. Let $\lambda \in W'$ be the Teichmüller lifting of λ_0. Let $e : A_\infty \to W'$ be the W-morphism sending t to λ. Then the base change

$$e_0^*\mathcal{M} := \mathcal{M} \otimes_{(A,e)} W'$$

with the induced map $e^*\phi$ is an F-crystal over W'. The crystal $e_0^*\mathcal{M}$ is called the *Teichmüller representative of the point* e_0. Similarly, if $\mathcal{M}$ is a Hodge F-crystal over A_∞, then the Teichmüller representative $e_0^*\mathcal{M}$ is a Hodge F-crystal over W'.

Example. As before, we consider the following simple situation.

(i) Let $\mathcal{X}_0$ be a smooth hypersurface in $\mathbb{P}^{n+1}$ over A_0. Then the relative crystalline cohomology $\mathcal{M} := H^n_{cris}(\mathcal{X}_0/(A_0/W))$ is an F-crystal over A_∞ (see [6], (2.4) for the precise meaning for this). Let $e_0 : A_0 \to k'$ be a k-morphism to a perfect field extension k' of k. Then the Teichmüller representative $e_0^*\mathcal{M}$ of e_0 is the crystalline cohomology $H^n_{cris}(X_0/W')$ of the hypersurface $X_0 := e_0^*\mathcal{X}_0$ over k'.

(ii) Suppose that $\mathcal{X}_0$ is the special fiber of a smooth hypersurface $\mathcal{X}$ in $\mathbb{P}^{n+1}$ over A. Let $\mathcal{X}_\infty$ be the base change of $\mathcal{X}$ from A to A_∞. Then we have a canonical identification

$$H^n_{dR}(\mathcal{X}_\infty/A_\infty) = H^n_{cris}(\mathcal{X}_0/(A_0/W)).$$

The Hodge to de Rham spectral sequence on $H^n_{dR}(\mathcal{X}/A_\infty)$ degenerates and it provides a filtration on $\mathcal{M}$. With these data, $\mathcal{M}$ is then a Hodge F-crystal over A_∞. The Teichmüller representitive $e_0^*\mathcal{M}$ of $e_0 : A_0 \to k'$ is identified with the de Rham cohomology $H^n_{dR}(e^*\mathcal{X}_\infty/W')$.

(c) Unit roots and horizontal sections

Let $e_0 : A_0 \to k'$ be a k-morphism to a perfect field extension k' of k. Let W' be the Witt ring of k'. Suppose $e_0(t) = \alpha_0$ and denote by $\alpha \in W'$ the Teichmüller lifting of α_0. On $W'[[t-\alpha]]$, we put the natural connection ∇ and choose a σ-linear endomorphism ϕ by setting $\phi(t) = t^p$.

Now let $\mathcal{M}$ be an F-crystal over A_∞. Then we can extend the (∇, ϕ)-structure on $\mathcal{M}$ to the completion $W'[[t-\alpha]] \otimes_{A_\infty} \mathcal{M}$ of $\mathcal{M}$ at $(t-\alpha)$ by combining the corresponding structure on $W'[[t-\alpha]]$ introduced above.

Theorem 2.1 ([5]) *We retain the above notations. Let $\mathcal{M}$ be a Hodge crystal over A_∞. Let $\bar{k}$ be an algebraic closure of k and $W(\bar{k})$ be the ring of Witt vectors of $\bar{k}$. Suppose that $\mathcal{M}/\operatorname{Fil}^1\mathcal{M}$ is of rank one and for every k-morphism $e_0 : A_0 \to \bar{k}$, the Teichmüller representative $e_0^*\mathcal{M}$ contains a unit-root subcrystal of rank one. Then there exists a (unique) unit-root sub-F-crystal U of $\mathcal{M}$ such that $\mathcal{M} = U \oplus \operatorname{Fil}^1\mathcal{M}$ as A_∞-modules. Suppose further that U is locally generated by u over A_∞. Write $\phi(u) = fu$ for some $f \in A_\infty^*$. Then we have:*

(i) *Let* $e_0 : A_0 \to k'$ *be a* k*-morphism to a perfect field extension* k' *of* k *with* $e_0(t) = \alpha_0$ *where* u *is defined. Let* α *be the Teichmüller lifting of* α_0*. Then there exists an* $F \in W'[[t-\alpha]]$ *such that* $v := F \cdot u \in W'[[t-\alpha]] \otimes_{A_\infty} \mathcal{M}$ *is* ∇*-horizontal and moreover, the quotient* F/F^ϕ *is in fact the expansion of an element in* A_∞*.*

(ii) *There exists* $c \in W(\bar{k})$ *(depending on* F*) such that* $c \cdot v \in W(\bar{k}) \otimes_W \mathcal{M}$ *is fixed by* ϕ *and consequently,* $f = cF/(cF)^\phi$*.*

Proof. See [5], Theorem 4.1 and 4.1.8, 4.1.9 in its proof. □

The above theorem says that if the crystal contains a unique unit-root crystal U of rank one, then one can detect the frobenius action ϕ on U by choosing a local solution at any (Teichmüller) point α. The existence of such a solution is guaranteed by (i). We formulate this in the following special case.

Corollary 2.2 *Suppose* $\mathcal{M}$ *satisfies the condition in the above theorem and we retain the notations therein. Take a Teichmüller point* $\alpha \in W$ *with* $\mathcal{H}(\alpha) \not\equiv 0 \pmod p$*. Let* $\{a_i\}_{i=1}^n$ *be a local bases around* α *of* $\mathcal{M}$ *over* A_∞ *such that* $\{a_i\}_{i=2}^n \subset \operatorname{Fil}^1 \mathcal{M}$*. Suppose there exists a unique non-trivial collection* $F_i \in W[[t-\alpha]], 1 \le i \le n$*, up to multiplication by a constant simultaneously, such that*

$$F_1 a_1 + \cdots + F_n a_n \in W[[t-\alpha]] \otimes_{A_\infty} \mathcal{M}$$

is ∇*-horizontal. Then there exists* $\gamma \in W(\bar{k})$ *such that*

$$f := \gamma \cdot (F_1/F_1^\phi)$$

is in fact an element of A_∞*. Moreover, suppose* k *is a finite field of cardinality* p^a *and let* $e_0 : A_0 \to k$ *with* $e_0(t) = \lambda_0$ *be a* k*-point. Let* λ *be the Teichmüller lifting of* λ_0*. Then*

$$\pi_{\lambda_0} = f(\lambda)^{1+\sigma+\cdots+\sigma^{a-1}}$$

is the unique p*-adic unit root of the linear endomorphism* ϕ^a *on the crystal* $e_0^*\mathcal{M}$ *over* W*.*

Proof. The element $a_1 + F_1^{-1}(F_2 a_2 + \cdots + F_n a_n)$ must be the element u in the above theorem. Putting $\gamma = c/c^\sigma$, where c is the constant in statement (ii) above, the assertions follow. □

Remark. In the computation of unit roots, we usually choose the different lifted frobenius ϕ' with $\phi'(t-\alpha) = (t-\alpha)^p$ to simplify the frobenius twist of F_1 above when we find a local horizontal section at α (see Theorems 3.3 and 4.3). For the relation between different lifted frobenius operators on an F-crystal $\mathcal{M}$, see [5], §1.

3 Example: an Apéry type family of elliptic curves

In this section, we study an interesting example of a family of elliptic curves E_t. The equation of this family is taken from [2], p.204. For the modular interpretation of this family, see op.cit. We call it an Apéry type family since the coefficients in the expansion F of the analytic solution to the associated Picard-Fuchs equation are of Apéry type. We shall see the close relation between the solution F and various invariants of E_t over positive characteristic. In particular, by applying the discussion in §2, we can write down the p-adic unit root for a (generic) member E_{λ_0} over a finite field in terms of F and its frobenius twist. This phenomenon is similar to the case of Legedre family (see [3], §6 (i) and [5], §8).

(a) Over $\mathbb{C}$

The Apéry type family of elliptic curves E_t over $\mathbb{C}$ with the parameter t discussed here is defined by the projective closure of the affine equation in the plane

$$y^2 + (1-3t)xy + t^2(1-t)y = x^3. \tag{2}$$

One computes that E_t has discriminant

$$\Delta = t^6(1-t)^3(1-9t),$$

and j-invariant

$$j = \frac{(1-3t)^3(1-9t+3t^2-3t^3)^3}{\Delta}.$$

Thus we shall regard E_t as a family over $S = \operatorname{Spec}\mathbb{C}[t][\Delta^{-1}]$.

The Picard-Fuchs differential operator associated to the differential

$$\omega = \frac{dx}{2y + (1-3t)x + t^2(1-t)} \in \mathrm{Fil}^1\, H^1_{dR}(E_t/S) \tag{3}$$

of the family is given ([11], §11 and [2]) by

$$\mathcal{L} = t(1-t)(1-9t)\frac{d^2}{dt^2} + (1-20t+27t^2)\frac{d}{dt} - (3-9t). \tag{4}$$

It has four regular singular points at zeros of Δ and ∞. There is a unique formal power series $F(t)$ with constant term 1 satisfying $\mathcal{L}F = 0$ (loc.cit.). It is given explicitly by

$$F(t) = \sum_{n=0}^{\infty} \chi(n)t^n \tag{5}$$

with

$$\chi(n) = \sum_{i=0}^{n} \binom{n}{i}^2 \binom{2i}{i}. \tag{6}$$

Let $\omega' = \nabla(\frac{d}{dt})\omega$, where ∇ is the Gauss-Manin connection on $H^1_{dR}(E_t/S)$. One checks that over $\hat{S} = \operatorname{Spec}\mathbb{C}((t))$, the element

$$v = t(t-1)(9t-1)\left[F\omega' - F'\omega\right] \in H^1_{dR}(E_t/\hat{S}) \tag{7}$$

is a horizontal section with respect to ∇.

(b) The Hasse invariant

Fix an odd prime p. We now consider the Apéry type family over a finite field k of characteristic p. The equation (2) is equivalent to

$$y^2 = P_t(x) := 4x^3 + \left[(1-3t)x + t^2(1-t)\right]^2 \tag{8}$$

via a standard change of variables. Let $H = H(t)$ be the coefficient of x^{p-1} in the expansion of $P_t(x)^{(p-1)/2}$. Then H, regarded as a polynomial over k, is the Hasse invariant of the family E_t.

Proposition 3.1 *Let $F^{<p}(t)$ be the truncation up to degree $(p-1)$ of the series $F(t)$ defined in (5). Then $H(t) \equiv F^{<p}(t) \pmod p$ for every odd prime p.*

Proof. Write $p = 2m+1$. Let $z = \frac{1}{\sqrt{x}}$ be a local parameter at the origin of the elliptic curve. We have

$$P_t^m = (4x^3)^m \left[1 + \frac{(1-3t)^2}{4}z^2 + \frac{t^2(1-t)(1-3t)}{2}z^4 + \frac{t^4(1-t)^2}{4}z^6\right]^m .$$

Note that $4^m = 2^{p-1} \equiv 1 \pmod p$. Thus, as elements in $k[t]$, we have

$$\begin{aligned}
&\text{the coeff. } H \text{ of } x^{p-1} = x^{2m} \text{ in } P_t(x)^m \\
= \;& \text{coeff. of } z^{p-1} \text{ in } \left[1 + \frac{(1-3t)^2}{4}z^2 + \frac{t^2(1-t)(1-3t)}{2}z^4 + \frac{t^4(1-t)^2}{4}z^6\right]^m \\
= \;& \text{coeff. of } z^{p-1} \text{ in } \left[1 + \frac{(1-3t)^2}{4}z^2 + \frac{t^2(1-t)(1-3t)}{2}z^4 + \frac{t^4(1-t)^2}{4}z^6\right]^{-\frac{1}{2}} .
\end{aligned}$$

Notice that $H(t)$ has degree $\leq (p-1)$ and constant term $\equiv 1 \pmod p$.

On the other hand, if we write the expansion of the invariant differential ω along z as

$$\begin{aligned}
\omega &= \frac{dx}{\sqrt{P_t(x)}} \\
&= \left[1 + \frac{(1-3t)^2}{4}z^2 + \frac{t^2(1-t)(1-3t)}{2}z^4 + \frac{t^4(1-t)^2}{4}z^6\right]^{-\frac{1}{2}} dz \\
&= \left(\sum_{n=1}^{\infty} a_n z^n\right)\frac{dz}{z},
\end{aligned}$$

then $\mathcal{L}a_n \equiv 0 \pmod{n}$ for all n ([7]; also [11], Theorem 2.3). One then checks that the equation $\mathcal{L}F = 0$ has only one polynomial solution with constant term 1 of degree $\leq (p-1)$ over k. Thus the assertion follows. □

(c) The invariant differential

We consider the family E_t defined over $R = \mathbb{Z}[t][\Delta^{-1}]$, where Δ is the discriminant. Consider the equation defining E_t in the homogeneous form

$$Q_t(X,Y,Z) := Y^2Z + (1-3t)XYZ + t^2(1-t)YZ^2 - X^3.$$

For any non-negative integer n, let b_{n+1} be the coefficient of $(XYZ)^n$ in Q_t^n. Then the formal differential

$$\xi = \left(\sum_{n=1}^{\infty} b_n \tau^n\right) \frac{d\tau}{\tau} \tag{9}$$

is the expansion of an element in $\mathrm{Fil}^1\, H^1_{dR}(E_t/R)$ with respect to some local parameter τ ([10], Theorem 1). One computes that b_{n+1} is equal to

$$\begin{aligned} b_{n+1} &= \sum_{r=0}^{[\frac{n}{3}]} \binom{n}{r}\binom{n-r}{r}\binom{n-2r}{r}(1-3t)^{n-3r}\left(-t^2(1-t)\right)^r \\ &= (1-3t)^n \sum_{r\geq 0} \binom{n}{3r}\frac{(3r)!}{(r!)^3}\left(\frac{-t^2(1-t)}{(1-3t)^3}\right)^r \\ &= (1-3t)^n \sum_{r\geq 0} \frac{\left(\frac{-n}{3}\right)_r \left(\frac{-n+1}{3}\right)_r \left(\frac{-n+2}{3}\right)_r}{(r!)^3}\left(\frac{27t^2(1-t)}{(1-3t)^3}\right)^r \\ &= (1-3t)^n {}_3F_2\left(\begin{matrix}\frac{-n}{3}, \frac{-n+1}{3}, \frac{-n+2}{3} \\ 1,1\end{matrix}; \frac{27t^2(1-t)}{(1-3t)^3}\right). \end{aligned}$$

Proposition 3.2 *Retaining the above notations, we have*

(i) *As elements in $H^1_{dR}(E_t/R)$, we have $\xi = c \cdot \omega$ for some non-zero $c \in \mathbb{Q}$, where ω is the invariant differential defined in equation (3).*

(ii) *$\mathcal{L}b_n \equiv 0 \pmod{n}$ for all positive integers n, where $\mathcal{L}$ is the Picard-Fuchs operator defined in equation (4).*

(iii) *As formal power series in x over $\mathbb{Z}$, we have*

$$F(x) = (1-3x)^{-1} {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3} \\ 1\end{matrix}; \frac{27x^2(1-x)}{(1-3x)^3}\right), \tag{10}$$

where $F(x)$ is the series defined in (5).

Proof. Let $\mathcal{L}'$ be the Picard-Fuchs differential operator associated to $\xi \in H^1_{dR}(E_t/R)$ with coefficients in $\mathbb{Z}[t]$. Then $\mathcal{L}' b_n \equiv 0 \pmod{n}$ for all n ([7]). Notice that the elliptic curve E_t over a field k of characteristic p is ordinary if and only if $b_p \neq 0$ in k. Now assume $p > 2$. Since b_p is a polynomial with constant term 1 and of degree $\leq (p-1)$, we have $b_p \equiv H \pmod{p}$, where H is the Hasse invariant. By varying p and applying Proposition 3.1, we see that $\mathcal{L}' = \alpha \mathcal{L}$ for some $\alpha \in \mathbb{Q}(t)$. Thus the assertions (i) and (ii) follow.

Take any prime p. When n runs through $n_s := p^s$, the sequence $b_n(x)$ converges p-adically to

$$B(x) = (1-3x)^{-1} {}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} ; \frac{27x^2(1-x)}{(1-3x)^3} \right).$$

Since $\mathcal{L} b_{n_s} \equiv 0 \pmod{p^n}$, we must have $B(x) = F(x)$. This proves (iii). $\square$

Remark.

(i) The identity (10) may have a modular interpretation.

(ii) For $t = 1/3$, the elliptic curve E_t over $\mathbb{Q}$ is equivalent to the one defined by the equation

$$y^2 = x^3 + 1.$$

Over $K = \mathbb{Q}(\sqrt{-3})$, the curve has complex multiplication by K.

(iii) One checks easily that

$${}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} ; 27z \right) \in 1 + 3z\mathbb{Z}[[z]].$$

Thus with $\chi(n)$ defined in (6), we have $3 \mid \chi(n)$ for all positive integers n by (10).

(d) The formula for the unit root

Fix an odd prime p. Let $\mathcal{H} = \Delta H$, where Δ is the discriminant and $H = F^{<p}(t)$. Let $A = \mathbb{Z}_p[t][\mathcal{H}^{-1}]$ and regard E_t as a family over A. With the series $F(t)$ given in (5), let

$$f(t) = F(t)/F(t^p),$$

regarded as a formal power series in t over $\mathbb{Z}_p$.

Theorem 3.3 *We retain the above notations. Let k be a finite field of characteristic $p \geq 5$ with cardinality p^a. For any $\lambda_0 \in k, \mathcal{H}(\lambda_0) \neq 0$, denote $\lambda \in W$ the Teichmüller lifting of λ_0. Then the series $f(t)$ converges p-adically at λ and the p-adic unit*

$$\pi_{\lambda_0} = f(\lambda)^{1+\sigma+\cdots\sigma^{a-1}} \tag{11}$$

is the unit root of the elliptic curve E_{λ_0} over k.

Proof. Since $p \neq 2$, the de Rham cohomology $H^1_{dR}(E_t/A)$ gives rise to a Hodge F-crystal (over A_∞ by base-change) satisfying the condition in Theorem 2.1. Since $p \neq 3$, the element v in equation (7) is the expression of the unique local horizontal section of the family up to multiplication by a constant ([5], Corollary 7.5). Thus the assertions follow by Corollary 2.2. The factor $\alpha(\lambda) := \lambda(\lambda-1)(9\lambda-1)$ in (7) does not appear in the formula (11) since

$$\left(\frac{\alpha(\lambda)}{\alpha(\lambda^p)}\right)^{1+\sigma+\cdots\sigma^{a-1}} = 1.$$

The fact that the constant $\gamma = 1$ in that corollary can be obtained by either applying [3], Lemma 6.2 combining the congruence $H \equiv F^{<p}(t) \pmod{p}$ or using the explicit expression of the invariant differential (9) with [11], Theorem (A.8) (v). $\square$

In testing how fast the series f converges p-adically, we have observed the following congruences numerically. It is an analogue of the congruences for coefficients of certain hypergeometric series proved by Dwork ([3], §1, Corollary 2). Notice that the equation (4) is not of hypergeometric type although the solution (5) is closely related to a hypergeometric series (see (10)).

Conjecture 3.4 *Let $\chi(n)$ be the integers defined in (6). Then*

$$\frac{\chi(\nu+\mu p+mp^{s+1})}{\chi(\mu+mp^s)} \equiv \frac{\chi(\nu+\mu p)}{\chi(\mu)} \pmod{p^{s+1}}$$

for all primes p and all non-negative integers ν, μ, m, s with $0 \leq \nu < p$.

4 Example: the Dwork family of Calabi-Yau threefolds

Here we give an example of families of Calabi-Yau threefolds, called the Dwork family, and study the variation of the unit root along this family. In the first part, we give a new interpretation of the Calabi-Yau condition of an order four ordinary linear differential equation in our setting. As in §3, we shall write the unit root of the member in this family in terms of the local solution to the Picard-Fuchs equation associated to this family. The Picard-Fuchs equation in this case is of hypergeometric type. In the latter part, we formulate some questions arising from the study of this family. For more details on the discussion in part (b) and (c) below and a generalization of the formula for the unit root along the Dwork family to the higher dimensional case, see [12].

(a) Calabi-Yau condition on differential equations of order four

Consider an ordinary linear differential operator of order four of the form

$$\mathcal{L} = \frac{d^4}{dt^4} + a_3\frac{d^3}{dt^3} + a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0, \quad a_i = a_i(t) \in \mathbb{Q}(t). \tag{12}$$

Recall the Calabi-Yau condition (among others) on the coefficients of $\mathcal{L}$ ([1], equation (2.2))

$$a_1 = a_2' + \frac{1}{2}a_2a_3 - \frac{1}{2}a_3'' - \frac{3}{4}a_3a_3' - \frac{1}{8}a_3^3. \tag{13}$$

For the meaning of this condition from the viewpoint of the theory of differential equations, see op.cit. §§2 and 3.

Lemma 4.1 *Let $\mathcal{L}$ be a differential equation of the form (12) satisfying the condition (13). Let (M, ∇) be a $\mathbb{Q}(t)/\mathbb{Q}$-differential module. Suppose $\omega \in M$ such that $\nabla(\mathcal{L})\omega = 0$. Write $\omega' = \nabla(\frac{d}{dt})\omega$, $\omega'' = \nabla(\frac{d}{dt})\omega'$ and so on. Let $F(t) \in \mathbb{Q}[[t]]$ be a formal solution to $\mathcal{L}F = 0$. Suppose there exists an element*

$$\alpha = \exp\left(\frac{1}{2}\int a_3\right) \in \mathbb{Q}[[t]]$$

(i.e., an element α satisfying $2\alpha' = a_3\alpha$). Then the section $u \in M \otimes_{\mathbb{Q}[t]} \mathbb{Q}[[t]]$ given by

$$\begin{aligned} v &= \alpha\left[F\omega''' - F'\omega'' + F''\omega' - F'''\omega\right] + (\alpha a_3 - \alpha')\left[F\omega'' - F''\omega\right] \\ &\quad + (\alpha a_2 - \alpha' a_3 - \alpha a_3' + \alpha'')\left[F\omega' - F'\omega\right] \end{aligned} \tag{14}$$

is horizontal with respect to ∇.

Proof. One shows that $\nabla(\frac{d}{dt})v = 0$ by direct computation. □

(b) The Dwork family over $\mathbb{C}$

Let V_t be the family of quintic hypersurfaces in $\mathbb{P}^4$ defined by the equation

$$P_t(X) := X_1^5 + \cdots + X_5^5 - 5tX_1 \cdots X_5. \tag{15}$$

We regard the family V_t as defined over $S = \operatorname{Spec} \mathbb{C}[t][(t(t^5-1))^{-1}]$. We remove the points $t^5 = 1$ since at each of those points, V_t is not smooth. The exclusion of $t = 0$ should be more transparent through the following discussion.

Let $\omega \in H^0(V_t/S, \Omega^3)$ be the residue of the meromorphic differential form

$$\frac{t \cdot \Omega}{P_t(X)} \quad \text{with } \Omega = \sum_{i=1}^{5} (-1)^i X_i dX_1 \wedge \cdots \wedge \widehat{dX_i} \wedge dX_5$$

on $\mathbb{P}^4$. Let $\tau = t^{-5}$ and $\theta = \tau\frac{d}{d\tau}$. We then check that ω satisfies

$$\nabla(\mathcal{L})\,\omega = 0,$$

where ∇ is the Gauss-Manin connection and

$$\mathcal{L} = \frac{d^4}{d\tau^4} + \frac{1}{\tau^3(1-\tau)}\left(2\tau^2(3-4\tau)\frac{d^3}{d\tau^3} + \tau(7 - \frac{72}{5}\tau)\frac{d^2}{d\tau^2} + (1 - \frac{24}{5}\tau)\frac{d}{d\tau} - \frac{24}{625}\right). \tag{16}$$

The operator $\mathcal{L}$ is of hypergeometric type. It has a unique formal power series solution at $\tau = 0$ with constant term 1 and it is given explicitly by the hypergeometric series

$$\begin{aligned} F(\tau) &= {}_4F_3\left(\begin{matrix} \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \\ 1, 1, 1 \end{matrix}\; ; \tau\right) \\ &= \sum_{r=0}^{\infty} \frac{(5r)!}{(r!)^5} \cdot \frac{\tau^r}{5^{5r}}. \end{aligned} \tag{17}$$

Notice that from the last expression, $F(\tau)$ has coefficients in $\mathbb{Z}[\frac{1}{5}]$.

The fact that $\mathcal{L}$ is of order four is reflected in that there is a direct summand $\mathcal{M}_{dR}$ of $H^3_{dR}(V_t/S)$ of rank four, which contains ω (see [4], Lemma 1.1). The decomposition of $H^3_{dR}(V_t/S)$ is stable under ∇. One checks that $\mathcal{L}$ satisfies the condition (13). In this case, the α-factor in Lemma 4.1 can be chosen to be

$$\begin{aligned}\alpha &= \exp\left(\int \frac{3-4\tau}{\tau(1-\tau)}\right) \\ &= \tau^3(1-\tau). \end{aligned} \tag{18}$$

Finally the element v in (14) with F in (17) and α in (18) is the unique (up to multiplication by a constant) local horizontal section near $\tau = 0$ with respect to ∇ ([4], Corollary 1.7).

(c) The Dwork family over a finite field

Now let k be a finite field of characteristic $p \neq 5$. Let $R = W[t][(t(t^5-1))^{-1}]$. Let $R_n = R/p^{n+1}R$ for any non-negative integer n, and $R_\infty = \varprojlim R_n$ be the projective limit. Consider the Dwork family V_t defined in (15) over R_n. Then as in the case over $\mathbb{C}$, there is a rank four direct summand $\mathcal{M}'$ of $H^3_{dR}(V_t/R_\infty)$ which contains $H^0(V_t/R, \Omega^3)$ and is stable under the Gauss-Manin connection ∇. The relative crystalline cohomology $H^3_{cris}(V_t/(R_0/W))$ of the family V_t over R_0 respects this decomposition and these data provide a Hodge F-crystal structure over R_∞ on $\mathcal{M}'$.

As in part (b), let $\tau = t^{-5}$. One can choose a generator $\omega \in H^0(V_t/R, \Omega^3)$ such that the Picard-Fuchs equation associated to ω is the same equation (16) as for the case over $\mathbb{C}$. With the formal power series solution $F(\tau) \in W[[\tau]]$ given in (17) to this equation, we let $H(\tau) = F^{<p}(\tau)$ be the truncation of the series up to degree $(p-1)$.

Proposition 4.2 *We retain the above notations. Then for any $\lambda_0 \in k, \lambda_0 \neq 0, \lambda_0^5 \neq 1$, the quintic V_{λ_0} has a (necessarily the unique) unit root if and only if $H(\lambda_0^{-5}) \neq 0$ in k. In other words, consider $\mathcal{H}(t) = t^{5p}(t^5-1)H(\tau)$ as a polynomial in t over W, where $\tau = t^{-5}$. Let $A = W[t][\mathcal{H}^{-1}]$, $A_n = A/p^{n+1}A$ and $A_\infty = \varprojlim A_n$. Then the crystal*

$$\mathcal{M} := \mathcal{M}' \otimes_{R_\infty} A_\infty$$

of rank four over A_∞ satisfies the condition in Theorem 2.1.

Proof. We can either directly counts the number of points in $V_{\lambda_0}(k)$ by Warning's method ([12], Theorem 4.2) or use the explicit realization ([10], Theorem 1) of the associated formal group of V_{λ_0} ([12], Lemma 3.3 (i)). $\square$

Theorem 4.3 *Let k be a finite field of characteristic $p \neq 5$ with cardinality p^a. Let W be the Witt ring of k. Let $F(\tau)$ be the hypergeometric series in (17) and let*

$$f(\tau) = F(\tau)/F(\tau^p) \tag{19}$$

as a formal power series in τ over W. For any $\lambda_0 \in k, \lambda_0 \neq 0, \lambda_0^5 \neq 1, H(\lambda_0^{-5}) \neq 0$, let $\lambda \in W$ be the Teichmüller lifting of λ_0. Then $f(\tau)$ converges at $\tau = \lambda^{-5}$ and the p-adic unit

$$\pi_{\lambda_0} = f(\lambda^{-5})^{1+\sigma+\cdots+\sigma^{a-1}} \tag{20}$$

is the unit root of the Calabi-Yau threefold V_{λ_0}.

Proof. By the discussion in part (b), one finds the expression of the unique local horizontal section of the crystal $\mathcal{M}$ constructed in the above proposition with respect to ∇. Thus the assertion follows by Corollary 2.2. The factor $\alpha(\lambda) := \lambda^{-15}(1-\lambda^{-5})$ in (14) given by (18) does not appear in the formula (20) since

$$\left(\frac{\alpha(\lambda)}{\alpha(\lambda^p)}\right)^{1+\sigma+\cdots\sigma^{a-1}} = 1.$$

Similar to the proof of Theorem 3.3, the constant $\gamma = 1$ in that corollary can be derived either by applying [3], Lemma 6.2 ([12], Theorem 4.3 (2)) or by using the explicit expression ([10], Theorem 1) of the associated formal group ([12], §5). $\square$

(d) Some open questions

In our attempt to write down a formula of a horizontal section for higher dimensional generalization of the family (15), we have observed the following identity, which we are not able to establish.

Conjecture 4.4 *For all positive integers $n > r$,*

$$\binom{n}{r} = \binom{n-r-1}{r} + \sum_{i=1}^{\left[\frac{r+1}{2}\right]} \left[\binom{r-i+1}{i} + \binom{r-i}{i-1}\right]\binom{n-r+i-1}{r-i}.$$

The following is some thought concerning the "canonical lifting" of a member in the Dwork family.

Let $\mathcal{M}$ be the Hodge F-crystal of rank four over A_∞ constructed in Proposition 4.2. Take a W-point λ of A_∞ and let $S = W[[t-\lambda]]$ be the formal completion of the parameter space at λ. Let $\mathcal{X} = V_t \times S$ be the base change of the family V_t to S and $\mathcal{N} = \mathcal{M} \otimes W[[t-\lambda]]$ be the associated crystal. Let $\mathcal{T}_{S/W}$ be the tangent space of S over W. Write $\mathrm{F}^i = \mathrm{Fil}^i\, H^3_{dR}(\mathcal{X}/S)$. Then we have the following commutative diagram

$$\begin{tikzcd} \mathcal{T}_{S/W} \arrow[r, "\kappa"] \arrow[dr, "\nabla"] \arrow[ddr, bend right, "\delta"'] & H^1(\mathcal{X}, \mathcal{T}_{\mathcal{X}/S}) \arrow[d, "\beta"] \\ & \mathrm{Hom}(\mathrm{F}^3, \mathrm{F}^2/\mathrm{F}^3) \arrow[d, "\gamma"] \\ & \mathrm{Hom}(\mathrm{F}^3, (\mathrm{F}^2 \cap \mathcal{N})/\mathrm{F}^3), \end{tikzcd}$$

where κ is the Kodaira-Spencer class associated to $\mathcal{X}/S$, the map β is induced from the natural pairing

$$H^1(\mathcal{X}, \mathcal{T}_{\mathcal{X}/S}) \times H^0(\mathcal{X}, \Omega^3) \to H^1(\mathcal{X}, \Omega^2),$$

and γ is the projection. Notice that δ is an isomorphism at least for almost all prime p since it is so over characteristic zero. Thus S may be regarded as the universal deformation space of $\mathcal{M}$ at λ.

On the other hand, fix a k-point of A_0 with $e_0(t) = \lambda_0$ and let $\lambda \in W$ be *any* lifting of λ_0. Let $X = V_{\lambda_0}$ over k and $Y = V_\lambda$ over W be the corresponding quintics. Let $M = e_0^* \mathcal{M}$. Suppose M is ordinary (see [12], Theorem 2.2). Then we have a slope decomposition $M = P \oplus P'$, where P and P' has slopes ≥ 2 and ≤ 2, respectively ([6], Theorem 1.6.1). The lifting Y gives rise to the relation

$$P \oplus P' = M \subset H^3_{cris}(X/W) = H^3_{dR}(Y/W) \supset \mathrm{Fil}^3_Y := H^0(Y/W, \Omega^3).$$

Let Q_Y be the projection of Fil^3_Y to P. Then Q_Y is an admissible filtration of P ([9], Definition (V.1.4)) and the pair (P, Q_Y) corresponds ([9], Theorem (V.1.6)) to a lifting G to W of the p-divisible group G_0 over k associated to P (more precisely, to the Tate twist $P(2)$). Together with the discussion in the previous paragraph, we get a bijection α from the set of liftings of X to W to the set of liftings of G_0 to W.

Notice that since G_0 is a direct sum of a 1-dimensional multiplicative formal group and an étale p-divisible group of rank 1, there is a "canonical lifting" of G_0 over W. This lifting thus corresponds to the "canonical lifting" of X to W.

Questions.

(i) Can one define the p-divisible groups G_0 and G in terms of cohomology of certain sheaves on the corresponding schemes? If this can be done, can one construct α functorially in terms of cohomology?

(ii) Suppose (i) can be done affirmatively. Is there any relation between the canonical lifting discussed above and the canonical coordinate q from mirror symmetry (see [8])? For any $\alpha_0 \in k$ such that V_{α_0} is ordinary, let α_{can} be the lifting of α_0 corresponding to the canonical lifting of V_{α_0}. Based on the work of Dwork (see [3], §7), we may also ask if there exists $\chi(t) \in t^p + pW[[t, t^{-1}]]$, such that $\chi^a(\alpha_{can}) = \alpha_{can}$ and the formal series $F(t)/F(\chi(t))$ has an analytic continuation to the domain strictly greater than the domain of $f(t)$ given in (19).

The questions above may be regarded as an attempt to generalize the Serre-Tate coordinate on the local moduli of an ordinary elliptic curve (see [9], Appendix) to a special motive of weight three and make a connection to the mirror map q mentioned in (ii) above.

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