On Ordinary Crystals with Logarithmic Poles[*]

Jeng-Daw Yu[†]

June 9, 2011

Abstract

We derive some local properties of abstract crystals with logarithmic poles over a smooth base in positive characteristic and obtain the existence of the canonical coordinates of certain ordinary crystals. We then apply the results to deduce an integral property of the coefficients of the so-called mirror maps.

Let k be an algebraically closed field of characteristic $p > 0$. The first aim of this paper is to investigate the abstract formalism of crystals over the ring $k[\![t_1, \cdots, t_m]\!]$ of formal power series over k with logarithmic poles (along some coordinate hyperplanes). We then derive some of the basic properties of the so-called ordinary crystals. (Precise definitions are given in the text.) Along the discussions, the definition and the existence of the (p-adic) canonical coordinates for an ordinary crystal over $\operatorname{Spec} k[\![t_i]\!]_{i=1}^m$ are obtained (see Thm 2.2). The existence of these canonical coordinates, which in some sense give a structural description of the ordinary crystal, is a generalization of a theorem of Deligne ([3, §1.4]) in the case without log structures. Indeed the development of these results is parallel to that of [3].

As an application, we derive that the coefficients of the mirror map $\tilde{q}$ of a nice family of Calabi-Yau varieties over a number field are p-adic integers for almost all prime numbers p. (See §3 for the precise statement.) The integral property of the coefficients of $\tilde{q}$ has been observed in the 1990s. In his letter to Morrison [4] in 1993, Deligne asked the question whether the coefficients of $\tilde{q}$ can be written algebraically in terms of the underlying family of Calabi-Yau varieties. We do not know if this is true or even how to formulate the statement properly. On the other hand, the integrality of $\tilde{q}$ was also observed independently in [5] around the same time. It has been proved in the first time in the paper [11] for certain hypergeometric cases using Dwork's criterion on the integrality of a p-adic formal power series. In fact, one of the main motivations of this work is the idea that the mirror map $\tilde{q}$ (defined via the variation of Hodge structure) should equal the canonical coordinates q (in the sense of Serre-Tate local coordinates studied here), and this relation should be revealed via the associated crystal. It turns out that the two q's are not exactly the same but they differ by a multiplicative p-adic unit (see Prop 3.2). However this suffices to deduce the integrality for $\tilde{q}$ (and also for the

[†]Department of Mathematics, National Taiwan University, Taipei, Taiwan; jdyu@math.ntu.edu.tw.

instanton numbers). Since the mirror maps are defined at the place where the fiber of the family is degenerate, one introduces the log structures to the family and to the crystal and this leads to the results in this study.

The method of using Dwork's criterion has been generalized in [10] to the mirror maps obtained from certain differential equations. This method has the advantage that it does not put constrains on the prime p and hence one obtains that the coefficients of the mirror maps are indeed integers. However to apply this method, one has to know explicitly the coefficients of the involved power series solutions to the differential equation. On the other hand, the use of ordinary crystals in the study of the mirror maps and the instanton numbers has already appeared in the work [15] in the non-singular case and has been outlined in [9] in the case with log poles. The case of rank 4 and weight 3 has been treated in [16] as a follow-up of [9]. There to get the comparison between the two q's mentioned above, the author uses Voevodsky's category of mixed motives and the theory of 1-motives to extract information from the weight ≤ 1 part of the rank four crystal, which behaves like a variation of 1-motives. However if the rank is larger, this method does not apply.

The paper is organized as follows. In the setup §1, we give the definitions of various crystals over the scheme $\operatorname{Spec} k[\![t_i]\!]_{i=1}^m$ equipped with the logarithmic structure attached to the union $(\prod_{i=1}^r t_i)$ of the hyperplanes $\{(t_i)\}_{i=1}^r$. In §2, we focus on ordinary crystals and derive the canonical coordinates of such a crystal (Thm 2.2), parallel to Deligne's theory in [3] for crystals with the trivial log structure. §3 consists of an application concerning the integrality of the mirror map (Prop 3.2) and of the instanton numbers (Thm 3.3) of a certain family of Calabi-Yau varieties. Finally in the appendix, we briefly discuss the group structure[1] on the underlying parameter space S associated with a certain ordinary crystal and the periods associated to the W-points $S(W)$.

Notations and conventions. Throughout this paper, we let $\mathbb{N} =$ the additive monoid of non-negative integers (with the unit 0). We use the convention $0^0 = 1$, the multiplicative unit.

We fix an algebraically closed field k of characteristic $p > 0$. Let $W =$ the ring of Witt vectors with coefficients in k and $K =$ the field of fractions of W. Denote by σ the absolute Frobenius on k, W and K. Our universal base for the crystalline cohomology theory of schemes over k is the scheme $\operatorname{Spec} W$ with the trivial logarithmic structure $W^\times$.

1 Crystals with logarithmic poles

In this section, we give the definitions of various crystals over $\operatorname{Spec} k[\![t_i]\!]_{i=1}^m$ and over $\operatorname{Spec} k$ with logarithmic poles and derive some relations among them. For the corresponding results of classical (i.e. without log poles) crystals, see [3, 8]. For definitions and more details regarding logarithmic structures, see [7].

---

[1]Thanks are due to C.-L. Chai for directing my attention to this question.

(a) The logarithmic schemes S and S_0

Throughout the paper[2], we fix $r, m \in \mathbb{N}$ with $r \leq m$. Let $t = \{t_i\}_{i=1}^m$ denote the m-tuple of variables t_i. Let $A = W[[t]]$ be the formal power series ring and

$$\mathcal{L} = \mathbb{N}^r \oplus A^\times \quad = \quad \bigcup_{n \in \mathbb{N}^r} t^n A^\times \tag{1}$$

$$(n, f) \quad \leftrightarrow \quad t^n f \tag{2}$$

where for $n = (n_i)_{i=1}^r \in \mathbb{N}^r$, we set $t^n = \prod_{i=1}^r t_i^{n_i}$ as usual. The monoid $\mathcal{L}$ is generated over $A^\times$ by $\{t_i\}_{i=1}^r$. The natural inclusion $\mathcal{L} \to A = \Gamma(\mathcal{O}, \mathrm{Spec}\, A)$ of monoids then defines a log structure on $\mathrm{Spec}\, A$. We will use the same symbol $\mathcal{L}$ to indicate the sheaf of monoids underlying this log structure; the sheaf $\mathcal{L}$ is coherent in the sense of [7, (2.1)]. Throughout the paper, let

$$S = (\mathrm{Spec}\, A, \mathcal{L})$$

denote this logarithmic scheme.

To describe the module of logarithmic differentials, which is generated by the log derivatives $d \log \mathcal{L}$ of the monoid $\mathcal{L}$, it is convenient to consider the twist $t' = \{t_i'\}_{i=1}^m$ of t giving by

$$t_i' = \begin{cases} t_i & \text{if } 1 \leq i \leq r \\ 1 + t_i & \text{if } r + 1 \leq i \leq m. \end{cases} \tag{3}$$

The A-module $\omega = \omega_{S/W}^1$ of logarithmic differential 1-forms of S over W is generated freely by

$$d \log t_i' = \frac{dt_i'}{t_i'} \quad (1 \leq i \leq m).$$

Consider the derivations

$$\delta_i = \frac{d}{dt_i'} \quad \text{and} \quad \theta_i = \frac{d}{d \log t_i'} = t_i' \frac{d}{dt_i'} \quad (1 \leq i \leq m). \tag{4}$$

The $\theta = \{\theta_i\}_{i=1}^m$ is dual to $d \log t'$. We have the following relations

$$(t_i')^n \delta_i^n = \prod_{j=0}^{n-1} (\theta_i - j) \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

Finally denote by

$$S_0 = (\mathrm{Spec}\, A_0, \mathcal{L}_0) \quad \text{and} \quad \omega_0 = \omega_{S_0/k}^1$$

the reductions mod p of S and ω, respectively. The ω_0 is the module of log differentials of the log scheme S_0 of characteristic p.

---

[2] The parameter r indicates the number of degenerating directions $\{t_i \in A\}_{i=1}^r$ on S, which produce the non-trivial logarithmic structure. When $r = 0$, one recovers the definitions in the classical situation where the involved log structure is the trivial one.

(b) Crystals over S_0

Let $\mathcal{H}$ be a free A-module. With the appearance of the log structure, a connection

$$\nabla : \mathcal{H} \to \omega \otimes_A \mathcal{H},$$

is called *quasi-nilpotent* (cf. [7, Remark (6.3)]) if for each $1 \le i \le m$, the endomorphism

$$\prod_{j=0}^{n-1} (\nabla(\theta_i) - j)$$

on $\mathcal{H}$ tends to zero p-adically as n goes to ∞. (Here θ_i is defined in (4).)

Definition. A *crystal* $(\mathcal{H}, \nabla)$ over S_0 is a pair of a free A-module $\mathcal{H}$ of finite rank equipped with an integrable connection

$$\nabla : \mathcal{H} \to \omega \otimes_A \mathcal{H},$$

which is quasi-nilpotent.

We now introduce the Frobenius and the Hodge structures on a crystal. The absolute Frobenius $\sigma(a) = a^p$ on A_0 induces an endomorphism on the sub-monoid $\mathcal{L}_0 \subset A_0$. We call this σ the *absolute Frobenius* on S_0 or on $(A_0, \mathcal{L}_0)$. Notice that a lifting $\phi : (A, \mathcal{L}) \to (A, \mathcal{L})$ of σ on $(A_0, \mathcal{L}_0)$ must have the form

$$\phi : t_i' \mapsto (t_i')^p f_i \quad (1 \le i \le m)$$

for some $f_i \in W[\![t]\!]^\times$ with $f_i \equiv 1 \pmod{p}$. (Here t_i' is defined in (3).)

Definition. An *F-crystal* $(\mathcal{H}, \nabla, \mathcal{F})$ over S_0 is a triple consists of a crystal $(\mathcal{H}, \nabla)$ over S_0 with a Frobenius structure $\mathcal{F}$, i.e. for any lifting ϕ on S of the absolute Frobenius, there is a horizontal A-linear map

$$\mathcal{F}(\phi) : \phi^* \mathcal{H} \to \mathcal{H},$$

which becomes an isomorphism after $\otimes_W K$. Here $\phi^* \mathcal{H} := A \otimes_\phi \mathcal{H}$. For $x \in \mathcal{H}$, we write $\phi^* x = 1 \otimes x \in \phi^* \mathcal{H}$.

If ϕ_1 and ϕ_2 are two liftings of σ, the compatibility of ∇ and $\mathcal{F}$ implies that there is a unique isomorphism $\chi(\phi_1, \phi_2)$ such that the following diagram commutes

$$\begin{array}{ccc} \phi_1^* \mathcal{H} & \xrightarrow{\ \mathcal{F}(\phi_1)\ } & \mathcal{H} \\ {\scriptstyle \chi(\phi_1,\phi_2)} \downarrow & \nearrow {\scriptstyle \mathcal{F}(\phi_2)} & \\ \phi_2^* \mathcal{H} & & \end{array} \qquad (6)$$

Indeed, if we write

$$\begin{aligned} \phi_1(t_i') &= (t_i')^p f_i \\ \phi_2(t_i') &= (t_i')^p g_i \end{aligned} \quad (f_i, g_i \in 1 + pW[\![t]\!], 1 \le i \le m),$$

the isomorphism $\chi(\phi_1, \phi_2)$ is given explicitly by

$$\chi(\phi_1, \phi_2)\phi_1^* x = \phi_2^* x + \sum_{|n|>0} \left(\frac{f}{g} - 1\right)^{[n]} \cdot \phi_2^* \left\{ \left(\prod_{1 \le i \le m}^{0 \le j \le n_i - 1} (\nabla(\theta_i) - j)\right) x \right\} \quad \text{for } x \in \mathcal{H}. \quad (7)$$

Here $f = (f_i)_{i=1}^m, g = (g_i)_{i=1}^m$ and the index of the summation runs through $n = (n_i) \in \mathbb{N}^m$ with $|n| := n_1 + \cdots + n_m > 0$. Notice that we have $f_i g_i^{-1} \equiv 1 \mod p$, and hence the divided powers $(\bullet)^{[n]} : pA \to pA$ in the above formula are meaningful.

Definition. A *Hodge F-crystal* $(\mathcal{H}, \nabla, \mathcal{F}, \mathrm{Fil}^\bullet)$ over S_0 consists of an *F-crystal* $(\mathcal{H}, \nabla, \mathcal{F})$ together with a decreasing filtration $\mathrm{Fil}^i \subset \mathcal{H}$ of free and co-free A-submodules indexed by $i \in \mathbb{N}$ such that

(i) $\mathrm{Fil}^0 = \mathcal{H}$ and $\mathrm{Fil}^n = 0$ for n sufficiently large,

(ii) $\nabla \mathrm{Fil}^{i+1} \subset \omega \otimes_A \mathrm{Fil}^i$ for all i, and

(iii) $\mathcal{F}(\phi)\phi^* \mathrm{Fil}^i \subset p^i \mathcal{H}$ for any lifting ϕ of the Frobenius and any i.

The *(Hodge) weight* of such an $\mathcal{H}$ is the sum $\rho_1 + \rho_2$ where

$$\rho_1 = \max\{n \in \mathbb{N} \mid \mathrm{Fil}^n = \mathcal{H}\};$$
$$\rho_2 = \max\{n \in \mathbb{N} \mid \mathrm{Fil}^n \ne 0\}.$$

(c) Crystals over logarithmic points

Consider the logarithmic scheme

$$\mathrm{Spec}\, W_{(r)} = (\mathrm{Spec}\, W, \mathbb{N}^r \oplus W^\times)$$

where the log structure is defined by the morphism of monoids

$$\mathbb{N}^r \oplus W^\times \rightarrow W$$
$$(n, x) \mapsto 0^n \cdot x.$$

The W-module $\omega_{(r)}$ of differential 1-forms is generated freely by the basis $\{\lambda_i\}_{i=1}^r$. Here if we let $\{\alpha_i\}_{i=1}^r$ be the standard basis of $\mathbb{N}^r$, then $\lambda_i = d\log(\alpha_i, 1)$, where $(\alpha_i, 1) \in \mathbb{N}^r \oplus W^\times$.

Let
$$\mathrm{Spec}\, k_{(r)} = (\mathrm{Spec}\, k, \mathbb{N}^r \oplus k^\times)$$

be the reduction of $\mathrm{Spec}\, W_{(r)} \mod p$. On the sheaf level, they fit into the commutative diagram

$$\begin{array}{ccc}
\mathbb{N}^r \oplus W^\times & \longrightarrow & W \\
{\scriptstyle (\mathrm{id},\, \mathrm{mod}\, p)} \downarrow & & \downarrow {\scriptstyle \mathrm{mod}\, p} \\
\mathbb{N}^r \oplus k^\times & \longrightarrow & k.
\end{array}$$

5

From the diagram, we see that there is a unique lifting, still call it σ, to $W_{(r)}$ of the absolute Frobenius on $k_{(r)}$ given by

$$\sigma : \mathbb{N}^r \oplus W^\times \rightarrow \mathbb{N}^r \oplus W^\times$$
$$(n, u) \mapsto (pn, u^\sigma).$$

Consequently we have $\lambda_i^\sigma = p\lambda_i$ for all i on the module $\omega_{(r)}$. We remark that the log scheme $\operatorname{Spec} W_{(r)}$ is the *canonical lifting* of $\operatorname{Spec} k_{(r)}$ in the sense of [6, Def (3.1)].

Definition. A *crystal* $(\mathcal{H}, \mathcal{N}_i)$ over $k_{(r)}$ is a pair consisting of a free W-module $\mathcal{H}$ of finite rank, endowed with a collection of pairwise commutative, quasi-nilpotent W-linear endomorphisms $\{\mathcal{N}_i\}_{i=1}^r$. An *F-crystal* $(\mathcal{H}, \mathcal{N}_i, \mathcal{F})$ over $k_{(r)}$ is a crystal $(\mathcal{H}, \mathcal{N}_i)$ together with a σ-linear map $\mathcal{F} : \mathcal{H} \rightarrow \mathcal{H}$, which becomes a bijection after $\otimes_W K$, and satisfies

$$\mathcal{N}_i \mathcal{F} = p\mathcal{F}\mathcal{N}_i \quad \text{for all } 1 \leq i \leq r. \tag{8}$$

One defines the notion of a *Hodge F-crystals* $(\mathcal{H}, \mathcal{N}_i, \mathcal{F}, \operatorname{Fil}^\bullet)$ over $k_{(r)}$ and its weight in a similar way as in the previous subsection (e.g., $\mathcal{N}_i \operatorname{Fil}^{j+1} \subset \operatorname{Fil}^j$, etc.).

Now we explain the relation between the above definition and the crystals obtained via connections. We let $\mathcal{C}$ denote the category of crystals over $k_{(r)}$.

Let $\mathcal{C}'$ be the category of pairs $(\mathcal{H}, \nabla)$ consisting of a free W-module $\mathcal{H}$ with a quasi-nilpotent, integrable connection

$$\nabla : \mathcal{H} \rightarrow \omega_{(r)} \otimes_W \mathcal{H}.$$

We have a natural functor

$$\mathcal{C} \rightarrow \mathcal{C}'$$
$$(\mathcal{H}, \mathcal{N}_i) \mapsto \left(\mathcal{H}, \nabla(h) := \sum_{i=1}^r \lambda_i \otimes \mathcal{N}_i(h) \right).$$

Here the commutativity of $\mathcal{N}_i$ translates to the integrability of ∇. The following assertion is clear.

Proposition 1.1 *The functor $C \rightarrow C'$ above establishes an equivalence of categories.*

(d) The Teichmüller representative

Fix an $l \in \mathbb{N}$. Any morphism $e_0 : (A_0, \mathcal{L}_0) \rightarrow k_{(l)}$ determines uniquely a pair of mophisms $(f_0, g_0) : \mathbb{N}^r \rightarrow \mathbb{N}^l \oplus k^\times$ of monoids obtained from the map between log structures

$$\mathcal{L}_0 = \mathbb{N}^r \oplus A_0^\times \rightarrow \mathbb{N}^l \oplus k^\times$$
$$(n, a(t)) \mapsto \big(f_0(n), g_0(n) \cdot a(0)\big)$$

with $f_0(n) = 0$ only if $n = 0$. Conversely any such a pair (f_0, g_0) determines a map e_0.

Lemma 1.2 *Let ϕ be a lifting to S of the Frobenius σ on S_0. For any $e_0 : (A_0, \mathcal{L}_0) \to k_{(l)}$, there exists a unique lifting $e = e(\phi) : (A, \mathcal{L}) \to W_{(l)}$ of e_0 such that the following diagram commutes*

$$
\begin{array}{ccc}
(A, \mathcal{L}) & \xrightarrow{\ e\ } & W_{(l)} \\
{\scriptstyle \phi} \downarrow & & \downarrow {\scriptstyle \sigma} \\
(A, \mathcal{L}) & \xrightarrow{\ e\ } & W_{(l)}.
\end{array}
\tag{9}
$$

Proof. Write $e = (e^\flat, e^\sharp)$, where $e^\flat : A \to W$ is a homomorphism and $e^\sharp : \mathcal{L} \to \mathbb{N}^l \oplus W^\times$ is compatible with $e^\flat$. The images $e^\flat(t_i)$ for $i > r$ are uniquely determined by ϕ as in the classical case (see [8, 1.1.2]).

Let $(f_0, g_0) : \mathbb{N}^r \to \mathbb{N}^l \oplus k^\times$ be the pair corresponding to e_0 as above. For $1 \leq i \leq r$, let $\alpha_i \in \mathbb{N}^r \subset \mathcal{L}$ corresponding to t_i via (1). In this case, we must have $e^\flat(t_i) = 0$ by a diagram chasing in (9). On the level of log structures, we have

$$
e^\sharp(\alpha_i, 1) = (f_0(\alpha_i), x)
$$

for some $x \in W^\times$ with

$$
x \equiv g_0(\alpha_i) \mod p. \tag{10}
$$

Write $\phi(t_i) = t_i^p \gamma$ for some $\gamma \in 1 + pA$. Then the commutativity of (9) implies that

$$
x^\sigma = x^p \cdot \gamma(0),
$$

which together with (10) determine x uniquely. $\qquad\square$

Definition. Given a lifting ϕ of the Frobenius on S, the (unique) lifting e, obtained in Lemma 1.2, of the morphism $e_0 : (A_0, \mathcal{L}_0) \to k_{(l)}$ is called the *Teichmüller lifting* of e_0 relative to ϕ.

The lifting e induces a map $\omega^1_{S/W} \to \omega^1_{W_{(l)}/W}$ between differentials. By the correspondence in Prop 1.1, the *pull-back* $e^*\mathcal{H} = \mathcal{H} \otimes_A W$ of a (resp. Hodge) F-crystal $\mathcal{H}$ over S_0 by the Teichmüller lifting, defines a (resp. Hodge) F-crystal over $k_{(l)}$. The following lemma shows that the pull-back of an F-crystal is independent of the lifting ϕ.

Lemma 1.3 *Let $\mathcal{H}$ be an F-crystal over S_0 and $e_0 : (A_0, \mathcal{L}_0) \to k_{(l)}$ a morphism. Suppose ϕ_1 and ϕ_2 are two liftings of the Frobenius and let $e(\phi_1)$ and $e(\phi_2)$ be the corresponding Teichmüller liftings. Then the connection on $\mathcal{H}$ provides a canonical identification $e(\phi_1)^*\mathcal{H} = e(\phi_2)^*\mathcal{H}$ as F-crystals over $k_{(l)}$.*

Proof. (Cf. [8, 1.4]) As in the proof of Lemma 1.2, we write

$$
\begin{aligned}
e(\phi_i)^\sharp(\alpha_j) &= (f_0(\alpha_j), \gamma_j(\phi_i)) & (1 \leq j \leq r) \\
e(\phi_i)^\flat(1 + t_j) &= \gamma_j(\phi_i) & (r < j \leq m)
\end{aligned}
$$

to represent $e(\phi_i)$ for $i = 1, 2$. Let $\gamma(\phi_i) = \{\gamma_j(\phi_i)\}_{j=1}^m$. Then the connection provides a canonical identification (cf. (7))

$$
e(\phi_1)^*\mathcal{H} \quad \cong \quad e(\phi_2)^*\mathcal{H} \tag{11}
$$

$$
e(\phi_1)^*\xi \quad \mapsto \quad \sum_{n \in \mathbb{N}^m} \left(\frac{\gamma(\phi_1)}{\gamma(\phi_2)} - 1 \right)^{[n]} e(\phi_2)^* \left\{ \left(\prod_{1 \leq i \leq m}^{0 \leq j \leq n_i - 1} (\nabla(\theta_i) - j) \right) \xi \right\} \quad \text{for } \xi \in \mathcal{H}.
$$

Let $\{\mathcal{N}_\mu(\phi_i)\}_{\mu=1}^l$ be the associated quasi-nilpotent operators on $e(\phi_i)^*\mathcal{H}$. Via Prop 1.1, they equal the pull-backs by $e(\phi_i)$ of the residues of ∇ on $\mathcal{H}$. Thus we have

$$\mathcal{N}_\mu(\phi_i)\big(e(\phi_i)^*\xi\big) = \sum_{\nu=1}^r \big[f_0(\alpha_\nu)\big]_\mu \cdot e(\phi_i)^*\big(\nabla(\theta_\nu)\xi\big) \quad (1 \le \mu \le l)$$

where $[\bullet]_\mu$ denotes the μ-th components of elements in $\mathbb{N}^l$. One computes trivially that $\mathcal{N}_\mu(\phi_1)(e(\phi_i)^*\xi) \mapsto \mathcal{N}_\mu(\phi_2)(e(\phi_i)^*\xi)$ under the identification (11) above.

On the other hand, the induced Frobenius structures on the two pull-backs are the same by applying e^* to the diagram (6). This completes the proof. □

Thus by the above lemma, we see that on $e^*\mathcal{H}$, the Frobenius structure and the $\mathcal{N}_j$ depend only on e_0, but not the choice of ϕ. We denote this F-crystal by $e_0^*\mathcal{H}$. We remark that the Hodge filtration on $e_0^*\mathcal{H}$ do depend on ϕ (cf. Prop A.2), but the Hodge polygon does not.

2 Ordinary crystals

(a) The setup

Definition. An F-crystal $(\mathcal{H}, \nabla, \mathcal{F})$ over S_0 is called a *unit-root F-crystal* if for some (and hence for all) ϕ of the liftings of the absolute Frobenius, $\mathcal{F}(\phi)$ is an isomorphism. One defines the notion of a *unit-root F-crystal over $k_{(r)}$* in a similar way.

Note that if $\mathcal{H}$ is a unit-root F-crystal over $k_{(r)}$, then $\mathcal{N}_j$ are all trivial. Indeed the equation (8) shows that each $\mathcal{N}_j$ is divisible by any power of p and hence is identically zero.

On the other hand, if $\mathcal{H}$ is a unit-root F-crystal over S_0, then there exists one basis $\{e_i\}_{i=1}^r$ of $\mathcal{H}$ over A such that $\nabla e_i = 0$ and $\mathcal{F}(\phi)\phi^*e_i = e_i$ for all liftings ϕ of the Frobenius. Thus $\mathcal{H}$ is indeed an F-crystal over A_0 (i.e. without poles). Indeed one first constructs a basis p-adically inductively verifying $\mathcal{F}(\phi)\phi^*e_i = e_i$ for a fixed lifting ϕ. Then $\{e_i\}$ satisfies the desired properties. The proof is identical to that of [3, Prop 1.2.2] after replacing $\Omega^1_{A/W}$ by ω (granted the fact that the log structure is irrelevant).

Definition.[3] Let $e_0 : S_0 \to \operatorname{Spec} k_{(r)}$ be the augmentation. A Hodge F-crystal $\mathcal{H}$ over S_0 is called an *ordinary crystal* if the Newton and the Hodge polygons of the F-crystal $e_0^*\mathcal{H}$ over $k_{(r)}$ coincide. (Cf. [3, Prop 1.3.2].)

Proposition 2.1 *Let $\mathcal{H}$ be an ordinary crystal over S_0. There exists a unique filtration of $\mathcal{H}$ by sub-F-crystals*

$$0 \subset \mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_i \subset \mathcal{U}_{i+1} \subset \cdots \tag{12}$$

which satisfies the following two properties:

---

[3] In [3], such a crystal in this definition is called an *ordinary Hodge F-crystal*. Here we shorten the terminology for abbreviation.

(i) $\mathcal{U}_i/\mathcal{U}_{i+1}$ is of the form $\mathcal{V}_i(-i)$, where $\mathcal{V}_i$ is a unit-root F-crystal and $(-i)$ denotes the Tate twist.

(ii) We have the decompositions of the A-module

$$\mathcal{H} = \mathcal{U}_i \oplus \mathrm{Fil}^{i+1} \quad \text{for all } i$$

and

$$\mathcal{H} = \bigoplus_{i \in \mathbb{N}} \mathcal{H}^{(i)} \quad \text{where } \mathcal{H}^{(i)} := \mathcal{U}_i \cap \mathrm{Fil}^i. \tag{13}$$

In particular, if $e_0 : S_0 \to \mathrm{Spec}\, k_{(r)}$ is the augmentation, then $\mathcal{N}_j^{\rho+1} = 0$ on $e_0^* \mathcal{H}$ for all j, where $\rho =$ the weight of $\mathcal{H}$.

Proof. The assertions (i) and (ii) are proved identically as in the classical case, see [3, §1.3]. Indeed one constructs the filtration $\mathcal{U}_\bullet$ inductively by using the Frobenius structure, which itself does not see the log structure, and reducing to the unit-root case where the log structure is irrelevant by the discussion above.

Since $\mathcal{N}_j$ acts trivially on the successive quotients $e_0^*(\mathcal{U}_i/\mathcal{U}_{i+1})$, we have $\mathcal{N}_j^{\rho+1} = 0$. $\square$

(b) Ordinary crystals of weight one

Let $\mathcal{H}$ be an ordinary crystal of weight one over S_0. Write $\mathcal{H} = \mathcal{U} \oplus \mathrm{Fil}^1$ as in Prop 2.1, where $\mathcal{U}$ is the unit-root part of $\mathcal{H}$. The following theorem describes the structure of $\mathcal{H}$; the proof will be given in the next subsection.

Theorem 2.2 (Cf. [3, Th 1.4.2]) *With assumptions as above, we have the following.*

(i) *There exist bases $a = \{a_i\}_{i=1}^g$ and $b = \{b_j\}_{j=1}^h$ of the A-modules $\mathcal{U}$ and Fil^1, respectively, verifying*

$$\begin{array}{rll} \nabla a_i &= 0 & (1 \leq i \leq g) \\ \nabla b_i &= \sum_{j=1}^g \eta_{ij} \otimes a_j & (1 \leq i \leq h), \end{array} \tag{14}$$

for some closed $\eta_{ij} \in \omega_{S/W}^1$, and for every lifting ϕ of the Frobenius,

$$\begin{array}{rll} \mathcal{F}(\phi)\phi^* a_i &= a_i & (1 \leq i \leq g) \\ \mathcal{F}(\phi)\phi^* b_i &= p b_i + p \sum_{j=1}^g u_{ij}(\phi) a_j & (1 \leq i \leq h), \end{array} \tag{15}$$

for some $u_{ij}(\phi) \in A$. Moreover η_{ij} and $u_{ij}(\phi)$ satisfy

$$\phi^* \eta_{ij} = p\eta_{ij} + p \cdot du_{ij}(\phi) \quad \text{for each } \phi. \tag{16}$$

(ii) *With the choice of a, b above, there exists a unique collection of elements*

$$\tau_{ij} = \tau_{ij}^{\log} + \tau_{ij}' \quad ((1,1) \leq (i,j) \leq (g,h))$$

with

$$(\tau_{ij}^{\log}, \tau_{ij}') \in \left(\bigoplus_{l=1}^r \mathbb{Z}_p \cdot \log t_l \right) \oplus K[\![t]\!]$$

9

such that for all i, j and ϕ,

$$\eta_{ij} = d\tau_{ij} \quad \text{with } \tau'_{ij}(0) \in pW \tag{17}$$

$$\phi^* \tau_{ij} - p\tau_{ij} - pu_{ij}(\phi) = 0. \tag{18}$$

(iii) *Suppose $p \neq 2$. The power series*

$$q'_{ij} = \exp(\tau'_{ij}) = \sum \frac{(\tau'_{ij})^n}{n!} \quad ((1,1) \leq (i,j) \leq (g,h)) \tag{19}$$

are well-defined elements in A, and verify $q'_{ij}(0) \equiv 1 \mod pW$.

Keep the $\mathcal{H} = \mathcal{U} \oplus \mathrm{Fil}^1$ over S_0 as above. The connection provides a linear map via the composition

$$\mathrm{Fil}^1 \xrightarrow{\nabla} \omega \otimes \mathrm{Fil}^0 \to \omega \otimes (\mathrm{Fil}^0 / \mathrm{Fil}^1) \xrightarrow{\sim} \omega \otimes \mathcal{U}.$$

Let $\omega^\vee_{S/W}$ denote the dual of the A-module ω. We then obtain an A-linear map

$$\mathrm{Gr}\, \nabla : \omega^\vee_{S/W} \to \mathrm{Hom}_A(\mathrm{Fil}^1, \mathcal{U}), \tag{20}$$

which in some sense measures the deviation of the crystal $\mathcal{H}$ over S_0. In a special case that will occur in the next section, we have the following.

Corollary 2.3 (Cf. [3, Cor 1.4.7]) *With the notations in Theorem 2.2, suppose p is odd, $g = 1$ and the map (20) is an isomorphism. Then we can further modify the basis $\{a, b\}$ of $\mathcal{H}$ above such that, in addition to the statements therein, the following hold :*

(i) *With q'_{1j} defined in (19), let*

$$q_j = \begin{cases} t_j q'_{1j} & \text{if } 1 \leq j \leq r \\ q'_{1j} & \text{if } r < j \leq m. \end{cases} \tag{21}$$

The W-homomorphism

$$(W[\![x]\!], \mathcal{L}_x) \quad \to \quad (A, \mathcal{L})$$

$$x_j \quad \mapsto \quad \begin{cases} q_j & \text{if } 1 \leq j \leq r \\ q_j - 1 & \text{if } r < j \leq m \end{cases}$$

is an isomorphism of affine log schemes, where $x = \{x_j\}_{j=1}^m$ and

$$\mathcal{L}_x = \bigcup_{n \in \mathbb{N}^r} x^n \cdot W[\![x]\!]^\times.$$

(ii) *Let ϕ be the lifting of the Frobenius defined by $\phi(q_i) = q_i^p$. We have*

$$\mathcal{F}(\phi)\phi^* b_i = pb_i \quad \text{for all } 1 \leq i \leq m.$$

Furthermore, the choice of $\{q_i\}_{i=1}^r$ depends only on the arrangement of $\{t_i\}_{i=1}^r$ modulo $(W[\![t_i]\!]_{i=r+1}^m)^\times$ multiplicatively.

Proof. Since (20) is a bijection, we can change the bases a and $\{b_j\}$ by multiplying by elements in $\mathbb{Z}_p^\times$ and $\mathrm{GL}_m(\mathbb{Z}_p)$, respectively such that $\tau_{1j}^{\log} = \log t_j$ for $1 \leq j \leq r$. Thus $\exp(\tau_{1j}) = t_j \exp(\tau'_{1j})$ are well-defined power series in A. (Note that the modification does not destroy the relations (14) and (15).) The statement (i) is now clear.

The assertion (ii) follows from (16) as in this case, $\phi^* \eta_{1j} = p\eta_{1j}$. $\qquad \square$

10

(c)   The proof of Theorem 2.2

To prove the assertions in (i), one reduces to the unit-root case, where the log structure is irrelevant, and thus can proceed identically as in the classical case (see [3, Th 1.4.2(i)] and replace $\Omega^1_{A/W}$ by ω there). Before the proof of (ii), we need the following lemma, which is the log analogue of the Poincaré lemma.

Lemma 2.4 *Let $x = \{x_i\}_{i=1}^g$ and $y = \{y_j\}_{j=1}^h$ be distinct variables and $\eta = \sum \alpha_i d\log x_i + \sum \beta_j dy_j$ be a closed differential form with $\alpha_i, \beta_j \in W[\![x,y]\!]$. We have $\eta = d\tau$ for a unique $\tau \in W \cdot \log x \oplus K[\![x,y]\!]$ up to an additive constant. Moreover $\tau \equiv \sum \alpha_i \log x_i \mod K[\![x,y]\!]$.*

Proof. By the assumption,

$$
0 = d\eta = \sum_{i,j} \left(x_i \frac{\partial \alpha_j}{\partial x_i} - x_j \frac{\partial \alpha_i}{\partial x_j} \right) \frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j} + \sum_{j,l} \left(x_j \frac{\partial \beta_l}{\partial x_j} - \frac{\partial \alpha_j}{\partial y_l} \right) \frac{dx_i}{x_i} \wedge dy_l
$$
$$
+ \sum_{l,n} \left(\frac{\partial \beta_n}{\partial y_l} - \frac{\partial \beta_l}{\partial y_n} \right) dy_l \wedge dy_n.
\tag{22}
$$

Pick any

$$
\tau_1 = \int \frac{\alpha_1}{x_1} dx_1 \in W \cdot \log x_1 \oplus K[\![x,y]\!].
$$

Notice that this is possible for the equation (22) above implies that $\alpha_1 - \alpha_1(0)$ is divisible by x_1 as an element in $W[\![x,y]\!]$. Let $\eta_1 = \eta - d\tau_1$. Then η_1 remains closed and

$$
\eta_1 = \sum_{i=2}^g \left(\alpha_i - x_i \frac{\partial \tau_1}{\partial x_i} \right) \frac{dx_i}{x_i} + \sum_{j=1}^h \left(\beta_j - \frac{\partial \tau_1}{\partial y_j} \right) dy_j
$$

does not have the term involving $\frac{dx_1}{x_1}$. Moreover, we have

$$
\frac{\partial}{\partial x_1} \left(\alpha_i - x_i \frac{\partial \tau_1}{\partial x_i} \right) = \frac{1}{x_1} \left(x_1 \frac{\partial \alpha_i}{\partial x_1} - x_i \frac{\partial \alpha_1}{\partial x_i} \right) = 0
$$

by (22) and similarly

$$
\frac{\partial}{\partial x_1} \left(\beta_j - \frac{\partial \tau_1}{\partial y_j} \right) = \frac{1}{x_1} \left(x_1 \frac{\partial \beta_j}{\partial x_1} - \frac{\partial \alpha_1}{\partial y_j} \right) = 0.
$$

Thus η_1 is independent of x_1. Now by induction and the classical Poincaré lemma (i.e. the case $g = 0$), the assertion follows. $\square$

Now back to the proof of Thm 2.2.

Proof of (17) — An inspection of (16) reveals that the residues of η_{ij} are in $\mathbb{Z}_p$. Together with Lemma 2.4, we then see that $\eta_{ij} = d\tau_{ij}$ for some $\tau_{ij} \in \mathbb{Z}_p \cdot \log t \oplus K[\![t]\!]$. We further write

$$
\begin{aligned}
\tau_{ij} &= \tau_{ij}^{\log} + \tau_{ij}' \\
&= \left(\sum_{l=1}^r \tau_{ij}^{(l)} \cdot \log t_l \right) + \tau_{ij}' \quad (\tau_{ij}^{(l)} \in \mathbb{Z}_p)
\end{aligned}
$$

11

corresponding to the direct sum decomposition as in the statement in (ii).

Recall the $t' = \{t'_l\}$ defined in (3). For any lifting ϕ, we have

$$\phi(t'_l) = (t'_l)^p f_l(\phi) \quad (1 \le l \le m)$$

for some

$$f(\phi) = (f_l(\phi)) \in \left(1 + pW[\![t]\!]\right)^m.$$

In particular

$$\log f_l(\phi) := -\sum_{n=1}^{\infty} \frac{(1 - f_l(\phi))^n}{n}$$

are well-defined elements in $pW[\![t]\!]$. Define $v_{ij}(\phi) \in W[\![t]\!]$ by letting

$$pv_{ij}(\phi) = pu_{ij}(\phi) - \sum_{l=1}^{r} \tau_{ij}^{(l)} \log f_l(\phi). \tag{23}$$

By substituting (23) into the integration of (16), we see that

$$\phi^* \tau'_{ij} - p\tau'_{ij} - pv_{ij}(\phi) \in K$$

is a constant and hence

$$\phi^* \tau'_{ij} - p\tau'_{ij} - pv_{ij}(\phi) = \left(\phi^* \tau'_{ij}\right)(0) - p\tau'_{ij}(0) - pv_{ij}(\phi)(0).$$

Now let ψ be the lifting given by sending t to t^p (and thus $v_{ij}(\psi) = u_{ij}(\psi)$). We normalize τ_{ij} by requiring

$$\left(\psi^* \tau'_{ij}\right)(0) - p\tau'_{ij}(0) - pv_{ij}(\psi)(0) = 0. \tag{24}$$

Since in this case, $\left(\psi^* \tau'_{ij}\right)(0) = \tau'_{ij}(0)^\sigma$, we have explicitly

$$\tau'_{ij}(0) = \sum_{n=1}^{\infty} V^n v_{ij}(\psi)(0) \quad \text{where } Vx := px^{\sigma^{-1}}.$$

This shows that $\tau'_{ij}(0) \in pW$.

Proof of (18) — It suffices to verify that for any lifting ϕ of the Frobenius, the equality (24) remains valid after replacing ψ by ϕ. We distinguish ϕ into two cases:

Case 1. The augmentation map $e : W_{(r)} \to S$ equals the Teichmüller representative relative to ϕ of the augmentation $e_0 = e \otimes_W k$. In this case,

$$f_l(\phi)(0) = 1 \quad \text{for all } 1 \le l \le m.$$

Thus $v_{ij}(\phi)(0) = u_{ij}(\phi)(0)$ and the two maps $\mathcal{F}(\phi)\phi^*$ and $\mathcal{F}(\psi)\psi^*$ induce the same Frobenius on $e^*\mathcal{H}$. Applying e^* to the the second equation in (15), we obtain

$$p(e^*b_i) + p\sum_{j=1}^{g} u_{ij}(\phi)(0)(e^*a_j) = \mathcal{F}(e^*b_i) = p(e^*b_i) + p\sum_{j=1}^{g} u_{ij}(\psi)(0)(e^*a_j).$$

12

Hence $u_{ij}(\phi)(0) = u_{ij}(\psi)(0)$.

Case 2. For general ϕ. We have

$$\phi(t'_l) = (t'_l)^p f'_l(\phi) \cdot (1 + pw_l^\sigma) \quad (1 \leq l \leq m)$$

for some $w = (w_l) \in W^m$ and $f'_l(\phi)(0) = 1$. Let ϕ_0 be the Frobenius with $\phi_0(t'_l) = (t'_l)^p f'_l(\phi)$. Then ϕ_0 is compatible with the augmentation e.

Applying $\chi(\phi, \phi_0)$ in (7) to the second equation for ϕ and ϕ_0 in (15), one obtains

$$p \sum_{j=1}^{g} u_{ij}(\phi) a_j = p \sum_{j=1}^{g} u_{ij}(\phi_0) a_j + \sum_{|n|>0} (pw^\sigma)^{[n]} \cdot \mathcal{F}(\phi_0) \phi_0^* \left\{ \left(\prod_{1 \leq l \leq m}^{0 \leq j \leq n_l - 1} (\nabla(\theta_l) - j) \right) b_i \right\}. \tag{25}$$

Using the second equation of (14) and plugging (17) in, one gets

$$\left(\prod_{1 \leq l \leq m}^{0 \leq j \leq n_l - 1} (\nabla(\theta_l) - j) \right) b_i = \sum_{j=1}^{g} (t')^n (\delta^n \tau_{ij}) a_j \quad (1 \leq i \leq h),$$

where $\delta_l = \frac{d}{dt'_l}$ (see Equation (5)). Thus, from (25) and the substitution (23), we deduce

$$pv_{ij}(\phi) = pv_{ij}(\phi_0) - \sum_{l=1}^{r} \tau_{ij}^{(l)} \log(1 + pw^\sigma) + \sum_{|n|>0} (pw^\sigma)^{[n]} \phi_0^* \left((t')^n \cdot \delta^n \tau_{ij} \right).$$

Therefore, to prove (24) for ϕ, it reduces to show that

$$\left(\phi^* \tau'_{ij} \right)(0) - p\tau'_{ij}(0) - pv_{ij}(\phi_0)(0)$$
$$+ \sum_{l=1}^{r} \tau_{ij}^{(l)} \log(1 + pw_l^\sigma) - \sum_{|n|>0} (pw^\sigma)^{[n]} \left((t')^n \cdot \delta^n \tau_{ij} \right)(0)^\sigma = 0.$$

For the last term, we have

$$\sum_{|n|>0} (pw^\sigma)^{[n]} \left((t')^n \cdot \delta^n \tau_{ij} \right)(0)^\sigma = \sum_{l=1}^{r} (\tau_{ij}^{(l)})^\sigma \sum_{n_l>0} (-1)^{n_l+1} (pw_l^\sigma)^{[n_l]}$$
$$= \sum_{l=1}^{r} \tau_{ij}^{(l)} \log(1 + pw_l^\sigma),$$

since $\tau_{ij}^{(l)} \in \mathbb{Z}_p$. Thus it suffices to show that

$$\left(\phi^* \tau'_{ij} \right)(0) - p\tau'_{ij}(0) - pv_{ij}(\phi_0)(0) = 0.$$

Notice that $\phi = \phi_0 \circ \alpha$, where α is the W-endomorphism of A defined by

$$\alpha(t'_l) = t'_l(1 + pw'_l) \quad \text{where } w'_l := \sigma^{-1}(w_l).$$

13

Hence we have

$$\left(\phi^*\tau'_{ij}\right)(0) = \left(\phi_0^*(\alpha^*\tau'_{ij})\right)(0) = \left(\alpha^*\tau'_{ij}\right)(0)^\sigma = \tau'_{ij}(0)^\sigma.$$

Granting the validity of (24) for ϕ_0, we have

$$\tau'_{ij}(0)^\sigma = p\tau'_{ij}(0) + pv_{ij}(\phi_0)(0).$$

Thus the equality (24) holds for ϕ.

The rest — The proof of the properties of q'_{ij} in (iii), which is a consequence of (18) and $\tau'_{ij}(0) \in pW$ by a trick of Dwork, is identical to that of [3, Th 1.4.2(ii)]. The proof of the theorem is now complete. $\square$

3 Ordinary crystals of Calabi-Yau type

(a) *The Calabi-Yau condition*

Definition. A Hodge F-crystal $(\mathcal{H}, \mathrm{Fil}^\bullet)$ of weight ρ over S_0 is called of *Calabi-Yau type* if Fil^ρ is of rank 1 over A and $\mathcal{H}$ is equipped with a horizontal, perfect, $(-1)^\rho$-symmetric pairing of F-crystals

$$\langle\,,\rangle : \mathcal{H} \times \mathcal{H} \to A(-\rho)$$

such that $\langle \mathrm{Fil}^i, \mathrm{Fil}^{\rho+1-i}\rangle = 0$ for all i.

Suppose $\mathcal{H}$ is ordinary and of Calabi-Yau type. Then it is easy to deduce from the definition that $\langle \mathcal{U}_i, \mathcal{U}_{\rho-1-i}\rangle = 0$ for all i, where $\mathcal{U}_\bullet$ is the filtration in (12). For the following theorem, cf. [3, Th 2.1.7], [4, §14] and [15, Thm 2.2].

Theorem 3.1 *Let $\mathcal{H}$ be an ordinary crystal of Calabi-Yau type of weight three over S_0. Suppose $p \neq 2$ and the map (20) is an isomorphism. Then there exist a basis $\{u_0, u_1^{(i)}, u_2^{(i)}, u_3\}_{i=1}^m$ of $\mathcal{H}$ with $\langle u_0, u_3\rangle = 1 = \langle u_1^{(i)}, u_2^{(i)}\rangle$, and elements $q_i \in A$ verifying the following properties:*

(i) *$A = W[\![q_i, q_j - 1]\!]_{1 \leq i \leq r}^{r < j \leq m}$ with*

$$\left.\begin{array}{cc} (t_i^{-1}q_i)(0) & (1 \leq i \leq r) \\ q_j(0) & (r < j \leq m) \end{array}\right\} \equiv 1 \mod pW.$$

(ii) *The connection is given by*

$$\begin{array}{rcl}
\nabla u_0 & = & 0 \\
\nabla u_1^{(i)} & = & d\log q_i \otimes u_0 \\
\nabla u_2^{(i)} & = & \sum_j \beta_{ij} \otimes u_1^{(j)} \qquad \textit{for some } \beta_{ij} \in \omega^1_{S/W} \\
\nabla u_3 & = & -\sum_i d\log q_i \otimes u_2^{(i)}.
\end{array}$$

14

(iii) *Write*

$$\beta_{ij} = \sum_l \kappa_{ijl} \cdot d\log q_l \quad (\kappa_{ijl} \in A). \tag{26}$$

If $\phi : A \to A$ is the lifting of the Frobenius given by $\phi(q_i) = q_i^p$, then

$$
\begin{aligned}
\mathcal{F}(\phi)\phi^* u_0 &= u_0 \\
\mathcal{F}(\phi)\phi^* u_1^{(i)} &= p u_1^{(i)} \\
\mathcal{F}(\phi)\phi^* u_2^{(i)} &= p^2 (a_i u_0 + \sum_j b_{ij} u_1^{(j)} + u_2^{(i)}) \\
\mathcal{F}(\phi)\phi^* u_3 &= p^3 (c u_0 + \sum_i a_i u_1^{(i)} + u_3)
\end{aligned}
$$

for some $a_i, b_{ij}, c \in A$. Let $\partial_i = q_i \frac{\partial}{\partial q_i}$. We have

$$
\begin{aligned}
\partial_i c &= -2 a_i \\
\partial_j a_i &= b_{ij} \\
\partial_l b_{ij} &= \phi(\kappa_{ijl}) - \kappa_{ijl}.
\end{aligned} \tag{27}
$$

Proof. Let $\mathcal{U}_i, \mathcal{H}_i$ be defined as in Prop 2.1. Then $\mathcal{U}_2$, with the induced filtration from Fil$^\bullet$, is ordinary of weight 1. Thus by Cor 2.3, there exist $u_0, u_1^{(i)}$ and q_i satisfying the desired conditions in (i), (ii) and (iii).

Since $\mathcal{U}_0^\perp = \mathcal{U}_2$ and $\mathcal{U}_1^\perp = \mathcal{U}_1$, we can find a basis $\{u_i\}$ of $\mathcal{H}_2$ such that $\langle u_1^{(i)}, u_i \rangle = 1$. Let

$$u_i' = u_i - \sum_{j \neq i} \langle u_1^{(j)}, u_i \rangle u_j.$$

Then $\langle u_1^{(j)}, u_i' \rangle = 0$ if $i \neq j$ and $\{u_i'\}$ forms a basis of $\mathcal{H}_2$. Since the product $\langle , \rangle$ induces a perfect pairing between $\mathcal{H}_1$ and $\mathcal{H}_2$ and $\langle \mathcal{H}_1, \mathcal{H}_1 \rangle = 0 = \langle \mathcal{H}_2, \mathcal{H}_2 \rangle$, the elements $\langle u_1^{(i)}, u_i' \rangle$ must be invertible. Thus the set

$$\left\{ u_2^{(i)} = \langle u_1^{(i)}, u_i' \rangle^{-1} \cdot u_i' \right\}$$

forms a basis of $\mathcal{H}_2$, satisfying $\langle u_1^{(i)}, u_2^{(j)} \rangle = \delta_{ij}$.

Take $\langle u_3 \rangle = \mathcal{H}_3$ with $\langle u_0, u_3 \rangle = 1$. By the transversality of the Hodge filtration and that $\nabla(\mathcal{U}_{i+1}/\mathcal{U}_i) = 0$, we have

$$
\begin{aligned}
\nabla u_2^{(i)} &= \sum_j \beta_{ij} \otimes u_1^{(j)} \\
\nabla u_3 &= \sum_i \gamma_i \otimes u_2^{(i)}
\end{aligned}
$$

for some $\beta_{ij}, \gamma_i \in \omega$. Differentiating the equation $\langle u_1^{(i)}, u_3 \rangle = 0$, we get $\gamma_i = -d\log q_i$.

On the other hand, since $\mathcal{U}_2$ is a sub-crystal, we can write

$$\mathcal{F} u_2^{(i)} = p^2 \left(a_i u_0 + \sum_j \left(b_{ij} u_1^{(j)} + e_{ij} u_2^{(j)} \right) \right).$$

15

Here and in the rest of the proof, we abbreviate $\mathcal{F}(\phi)\phi^*$ as $\mathcal{F}$. Applying $\mathcal{F}$ to $\langle u_1^{(i)}, u_2^{(j)}\rangle = \delta_{ij}$, one gets $e_{ij} = \delta_{ij}$.

Suppose
$$\mathcal{F}u_3 = p^3 \left(cu_0 + \sum_i \left(f_i u_1^{(i)} + e_i u_2^{(i)} \right) + f u_3 \right).$$

Applying $\mathcal{F}$ to $\langle u_0, u_3\rangle = 1, \langle u_1^{(i)}, u_3\rangle = 0$ and $\langle u_2^{(i)}, u_3\rangle = 0$, one gets $f = 1, e_i = 1$ and $f_i = a_i$, respectively.

The relations (27) between a, b, c, β are derived from the flatness $\nabla\mathcal{F} = \mathcal{F}\nabla$. Applying it to u_3 and $u_2^{(i)}$, one obtains the first one and the last two equations, respectively. $\qquad\square$

Remark. For a discussion about the possible relationship between the constant term $c(0)$ of c above and a certain entry of the (complex) monodromy matrix of the lifted algebraic family, see [14] where the case of a family of quintic threefolds has been computed explicitly. The results obtained here do not provide any information of $c(0)$.

(b)   Crystals and the mirror maps from degenerating families

We now consider the situation where the crystals are obtained from certain families with a special kind of mild degenerations. Given a flat projective scheme X over $\operatorname{Spec} A$ with X smooth over W, we say that X/A is a *degenerating family of split type (along the directions $t_i, 1 \leq i \leq r$)* if locally there exist elements $\{x_{ij}\}_{1 \leq i \leq r}^{1 \leq j \leq n_i}$ in $\mathcal{O}_X$ for some integers $n_i > 1$, which, together with p, form a part of a regular sequence, such that $\mathcal{O}_A \to \mathcal{O}_X$ is given locally by
$$t_i \mapsto \prod_{j=1}^{n_i} x_{ij} \quad (1 \leq i \leq r).$$

Thus the fiber Y over $t_1 \cdots t_r = 0$ is a (reduced) normal crossing divisor of X, and locally every irreducible component of the fiber over any $t_i = 0, 1 \leq i \leq m$, is also of split type.

In this case, we equip X and A with the log structures attached to the divisors Y and $t_1 \cdots t_r = 0$, respectively. Then X/S is log-smooth; the collection $\{x_{ij}^{-1}dx_{ij}\}$ form a part of a local basis of the sheaf of log differential 1-forms on X and we have
$$\begin{aligned} \omega_S^1 &\to \omega_X^1 \\ \frac{dt_i}{t_i} &\mapsto \sum_{j=1}^{n_i} \frac{dx_{ij}}{x_{ij}} \quad (1 \leq i \leq r). \end{aligned}$$

Let X_0/S_0 be the reduction mod p. We have $H_{dR}^\rho(X/S) = H_{cris}^\rho(X_0/W)$, where the cohomologies are understood to be the logarithmic ones. Denote by $\omega_{X/S}^i$ the sheaf on X of relative log differential i-forms. Now suppose, for all i, j and ρ, that $H_{dR}^\rho(X/S)$ and $H^j(X, \omega_{X/S}^i)$ are free A-modules and the Hodge to de Rham spectral sequence
$$E_1^{ij} = H^j(X, \omega_{X/S}^i) \Longrightarrow H_{dR}^{i+j}(X/S)$$

degenerates at E_1 (see [7, Thm (4.12)(3)]). Then $H_{dR}^\rho(X/S)$ is a Hodge F-crystal of weight ρ over S_0 provided that $p > \rho$ by the log analogue of a theorem of Mazur and Ogus ([13, Cor 8.3.3] and cf. [2, Thm 8.26]).

Now suppose that K can be embedded into the field $\mathbb{C}$ of complex numbers and we fix one such embedding. Assume that

- $r = m$,

- the map (20) is an isomorphism, and

- the crystal $H^\rho_{dR}(X/S)$ is of Calabi-Yau type.

We define the *mirror maps* $\tilde{q} = \{\tilde{q}_i\}^m_{i=1}$ as follows.

Denote by $\tilde{u}_0$ the image of $u_0 \in H^\rho_{dR}(X/S)$ in

$$\mathcal{H}_\mathbb{C} = H^\rho_{dR}((X/S) \otimes_W \mathbb{C}) = H^\rho_{dR}(X/S) \otimes_W \mathbb{C}.$$

Let $\tilde{u}_1^{(1)}, \cdots, \tilde{u}_1^{(m)}$ be elements in the Hodge filtration $\widetilde{\mathrm{Fil}}^1$ of $\mathcal{H}_\mathbb{C}$ satisfying

$$\nabla(\theta_i)\tilde{u}_1^{(j)}\Big|_{t=0} = \delta_{ij} \cdot \tilde{u}_0\Big|_{t=0} \quad \text{for all } 0 \leq i, j \leq m, \tag{28}$$

where $\delta =$ Kronecker's delta. We define $\tilde{q}_i \in t_i(1 + t_i\mathbb{C}[\![t]\!]^\times)$ by requiring

$$\nabla\tilde{u}_1^{(i)} = d\log \tilde{q}_i \otimes \tilde{u}_0 \quad (1 \leq i \leq m).$$

We remark that the elements $\tilde{q}_i$ can also be defined via the Picard-Fuchs differential equation associated with $\mathcal{H}_\mathbb{C}$, cf. [17, §1].

Proposition 3.2 *With notations and assumptions as above, assume that the Hodge F-crystal $H^\rho_{dR}(X/S)$ is ordinary. Via the inclusions $W \subset K \subset \mathbb{C}$, we have $\tilde{q}_i \in W[\![t]\!]$ for all i.*

Proof. We compare the $\tilde{q}_i$ with the $q_i \in W[\![t]\!]$ defined in Thm 3.1. The condition (28) forces that $\tilde{q}_i = a_i q_i$ for some constant $a_i \in \mathbb{C}$. Since $t_i^{-1}\tilde{q}_i(0) = 1$ by definition, we have $a_i = (t_i^{-1}q_i)(0) \in W^\times$ and the assertion follows. $\quad\square$

(c) Global case

We now turn to the following global consideration. Let R be an open part of the integral closure of a number field $R_\mathbb{Q}$. Suppose that we have a degenerating family $\pi : X \to \mathrm{Spec}\, R[\![t]\!]$ of split type along $t_1, \cdots, t_m$ (defined in the similar way as in the beginning of §3(b)). As before we equip the family with the log structures associated with the degenerate fiber of X and the coordinate hyperplanes of the base $R[\![t]\!]$, respectively. Let $\tilde{\omega}$ denote the sheaf on X of the relative log differential 1-forms of $X/R[\![t]\!]$. We assume that $H^j(X, \tilde{\omega}^i)$ are free over $R[\![t]\!]$ for all i, j, and the relative Hodge to de Rham spectral sequence degenerate at E_1. These two requirements can be achieved by shrinking R.

Consider the relative de Rham cohomology

$$\widetilde{\mathcal{H}} = H^3_{dR}(X/R[\![t]\!])$$

of degree three with the Hodge filtration $\widetilde{\mathrm{Fil}}^\bullet$ and the Gauss-Manin connection ∇. Similar to (20), we have an $R[\![t]\!]$-linear map

$$\gamma : \omega^\vee_{R[\![t]\!]} \to \mathrm{Hom}_{R[\![t]\!]}(\widetilde{\mathrm{Fil}}^3, \widetilde{\mathrm{Fil}}^2/\widetilde{\mathrm{Fil}}^3), \qquad (29)$$

where $\omega_{R[\![t]\!]}$ denotes the module of log differential 1-forms of $R[\![t]\!]$. We assume that the map γ is an isomorphism and $\widetilde{\mathcal{H}}$ is of Calabi-Yau type (defined in a similar way as in the beginning of §3(a)). Just as in the constructions of previous two subsections, there exists a basis $\{\tilde{u}_0, \tilde{u}_1^{(i)}, \tilde{u}_2^{(i)}, \tilde{u}_3\}_{i=1}^m$ of $\widetilde{\mathcal{H}}$ and elements $\tilde{q} = \{\tilde{q}_i\}_{i=1}^m$ and $\{\tilde{\kappa}_{ijk}\}_{1 \le i,j,k \le m}$ of $R[\![t]\!]$ satisfying the following three conditions (cf. [4, §14] for the considerations from the point of view of the Betti side):

(i) $\tilde{u}_0$ generates the unique rank one R-module of horizontal sections in $\widetilde{\mathcal{H}}$.

(ii) We have $\tilde{q}_i \in t_i(1 + t_i R[\![t]\!]^\times)$ and $\nabla \tilde{u}_1^{(i)} = d\log \tilde{q}_i \otimes \tilde{u}_0$ for $1 \le i \le m$ as in the previous subsection.

(iii) Together with $\{\tilde{u}_0, \tilde{u}_1^{(i)}\}_{i=1}^m$, the reminder $\{\tilde{u}_2^{(i)}, \tilde{u}_3\}_{i=1}^m$ form a symplectic basis of $\widetilde{\mathcal{H}}$ similar to the description in Thm 3.1. Define similarly

$$\tilde{\kappa}_{ijl} \in R[\![t]\!] = R[\![\tilde{q}]\!]$$

corresponding to the elements κ_{ijl} given in (26).

For a prime $\mathfrak{p}$ of R, denote by $R_\mathfrak{p}$ its completion at $\mathfrak{p}$ and by $X_\mathfrak{p}/R_\mathfrak{p}[\![t]\!]$ the base change of $X/R[\![t]\!]$. Consider the crystal

$$\mathcal{H}_\mathfrak{p} = H^3_{dR}(X_\mathfrak{p}/R_\mathfrak{p}[\![t]\!]) = \widetilde{\mathcal{H}} \otimes_R R_\mathfrak{p}.$$

Let $e : R_\mathfrak{p}[\![t]\!] \to R_\mathfrak{p}$ be the augmentation. Denote by $\bar{u}_0 \in e^* \mathcal{H}_\mathfrak{p}$ the induced element of $\tilde{u}_0$ through the base changes. As an application of previous results, we have the following.

Theorem 3.3 *With notations and assumptions as above, suppose that for each prime $\mathfrak{p}$ of R, the crystal $\mathcal{H}_\mathfrak{p}$ is ordinary and $\bar{u}_0$ and $(t_i^{-1} q_i)(0), 1 \le i \le m$, are fixed by the absolute Frobenius, where q_i are defined in Theorem 3.1. Consider the expansion*

$$\tilde{\kappa}_{ijl} = \varepsilon_{ijl}(0) + \sum_{|n|>0} \varepsilon_{ijl}(n)\frac{n_i n_j n_l \tilde{q}^n}{1 - \tilde{q}^n} \quad (1 \le i,j,l \le m, \varepsilon_{ijl}(n) \in R_\mathbb{Q}).$$

Then $\varepsilon_{ijl}(0) \in R \cap \mathbb{Q}$ and $\mathrm{tr}(\varepsilon_{ijl}(n)) \in \mathrm{tr}\, R$ for all $n \ne 0$, where $\mathrm{tr} : R_\mathbb{Q} \to \mathbb{Q}$ is the trace map.[4]

Proof. Indeed, on $\mathcal{H}_\mathfrak{p}$, the two bases $\{\tilde{u}_0, \tilde{u}_1^{(i)}, \tilde{u}_2^{(i)}, \tilde{u}_3\}$, induced from that of $\widetilde{\mathcal{H}}$ by the natural base change, and $\{u_0, u_1^{(i)}, u_2^{(i)}, u_3\}$, constructed in Thm 3.1, only differ by a multiplicative p-adic integer. More precisely, let W be the ring of Witt vectors of an algebraic closure of $R/\mathfrak{p}$. Then there exists a non-zero $\alpha \in W$ such that $u_0 = \alpha \tilde{u}_0$. Since

---

[4] The $\varepsilon_{ijl}(n)$ are the *instanton numbers*.

$\mathcal{F}\bar{u}_0 = \bar{u}_0$, we obtain $\alpha \in \mathbb{Z}_p$. Consequently we have $u_1^{(i)} = \alpha \tilde{u}_1^{(i)}$ and $u_2^{(i)} = \alpha^{-1}\tilde{u}_2^{(i)}$ for all i. Thus $\kappa_{ijl} = \alpha^2 \tilde{\kappa}_{ijl}$. On the other hand, we have $R_{\mathfrak{p}}[\![\tilde{q}]\!] = R_{\mathfrak{p}}[\![q]\!]$ and $\tilde{q}_i \frac{\partial}{\partial \tilde{q}_i} = q_i \frac{\partial}{\partial q_i}$ by the proof of Prop 3.2. The assertion then follows from Thm 3.1 above and Lemma 3.4 below. (Note that if the Frobenius sends q_i to q_i^p, then it sends $\tilde{q}_i$ to $\tilde{q}_i^p$ by our assumption on the leading terms of q_i.) $\qquad\square$

Lemma 3.4 ([9, Lemma 1, 2]) *Let V be the ring of Witt vectors of a finite field of characteristic p and $\mathrm{tr} : \mathbb{Q} \otimes_{\mathbb{Z}} V \to \mathbb{Q}_p$ the trace map. Fix $1 \le i_j \le m$ for $j = 1, \cdots, r$. Let $\kappa \in V[\![q]\!]$ be written formally as*

$$\kappa = \varepsilon_0 + \sum_{|n|>0} \varepsilon_n \frac{n_{i_1} n_{i_2} \cdots n_{i_r} q^n}{1 - q^n} \quad (n = (n_l)_{l=1}^m \in \mathbb{N}^m, \varepsilon_n \in \mathbb{Q} \otimes_{\mathbb{Z}} V).$$

Let ϕ be the lifting of the absolute Frobenius on $V[\![q]\!]$ given by $\phi(q) = q^p$ and $\partial_i = q_i \frac{\partial}{\partial q_i}$. Suppose

$$\phi(\kappa) - \kappa = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_r} f \quad \text{for some } f \in V[\![q]\!]. \tag{30}$$

Then

$$\mathrm{tr}\, \varepsilon_n \in \mathbb{Z}_p \quad \text{for all } n \ne 0.$$

Proof. Let s be the rank of V over $\mathbb{Z}_p$. The assertion in the case $(r, s) = (3, 1)$ is proved by using the Möbius inversion formula in [9, §3] (see especially Equation (30) therein). The general case is proved similarly. We remark that in case $s = 1$, the condition (30) is also necessary. $\qquad\square$

Remarks. (i) One know that if the irreducible components of the central fiber of the family $X_{\mathfrak{p}}/R_{\mathfrak{p}}[\![t]\!]$ are ordinary, the crystals $H_{dR}^\rho(X_{\mathfrak{p}}/R_{\mathfrak{p}}[\![t]\!])$ are ordinary ([12, 3.23]). With notations as above, it is easily seen that the cohomology group $\widetilde{\mathcal{H}}$ is of Hodge-Tate type in the sense of [4, §6] if $\mathcal{H}_{\mathfrak{p}}$ is ordinary for some $\mathfrak{p}$ by Prop 2.1.

(ii) The conditions on $\bar{u}_0$ and $(t_i^{-1} q_i)(0)$ in Thm 3.3 can be checked by looking at the F-crystal $e^* \mathcal{H}_{\mathfrak{p}}$. On one hand, $\mathcal{F}\bar{u}_0$ is a constant multiple of $\bar{u}_0$; on the other hand, the leading terms of q_i are normalized by (18) (see also (24)). However, the log structure on $e^* X$ depends on its embedding to X.

A   Appendix: The group structure

In the classical case, the deformation functor to local W-algebras of an ordinary abelian variety or K3 surface X over the field k has a natural formal group structure, and this leads to the notion of the canonical lifting of X (see [3]). In this appendix, we provide an analogous group structure on the deformation of an ordinary crystal of weight one under the appearance of a logarithmic structure. The explanation here is not satisfactory yet since the group functor is not (pro-)representable in this case.

Fix $r \in \mathbb{N}$. In this section, let $\{e_i\}_{i=1}^r$ be the standard basis of $\mathbb{N}^r$.

Let $\widetilde{\mathfrak{Art}}$ be the category of Artinian local $\mathbb{Z}_p$-algebras $(R, \mathfrak{m})$ with fixed inclusions $R/\mathfrak{m} \hookrightarrow k$, equipped with log structures $\mathcal{M}_R$ and structure morphims sitting in the sequence

$$(\mathbb{Z}_p, \mathbb{Z}_p^\times) \to (R, \mathcal{M}_R) \to k_{(r)} \tag{31}$$

of log rings. The morphisms of $\widetilde{\mathfrak{Art}}$ are the morphisms of log rings compatible with the structure morphisms (31).

For an Artinian local W-algebra $(R, \mathfrak{m})$ with a fixed $R/\mathfrak{m} \to k$, denote by $R_{(r)}$ the ring R equipped with the log structure

$$
\begin{aligned}
\mathbb{N}^r \oplus R^\times &\to R \\
(n, x) &\mapsto 0^n \cdot x.
\end{aligned}
\tag{32}
$$

and with the structure morphism $(\mathrm{id}, \mathrm{mod}\,\mathfrak{m}) : \mathbb{N}^r \oplus R^\times \to \mathbb{N}^r \oplus k^\times$. (We also extend this symbol to pro-Artinian local W-algebras.) Let $\mathfrak{Art}$ be the subcategory of $\widetilde{\mathfrak{Art}}$ consisting of such Artinian $R_{(r)}$'s whose morphisms consist of those which preserve the direct sums (32) of the log structures. Notice that $\mathfrak{Art}$ does not have an initial object. However each $R_{(r)}$ admits a morphism $(\mathbb{Z}_p)_{(r)} \to R_{(r)}$ whose log part $\mathbb{N}^r \oplus \mathbb{Z}_p^\times \to \mathbb{N}^r \oplus R^\times$ is the direct sum of the identity and the structure morphism. We denote this special object (with the arrows) by $\mathring{\mathbb{Z}}_p$.

(a)   The group functor $\mathbb{S}$

We define functors $\mathbb{G}, \mathbb{T}$ and $\mathbb{S}$ on $\mathfrak{Art}$, which are pro-representible in $\widetilde{\mathfrak{Art}}$. The functor $\mathbb{S}$ will be an extension of $\mathbb{T}$ by $\mathbb{G}$ and will provide the group structure attached to a certain ordinary crystal on the log scheme $S = (\mathrm{Spec}\,A, \mathcal{L})$ discussed in the following subsection.

Let $\mathbb{G} = \mathrm{Spf}\,\mathbb{Z}_p[\![z]\!]$, $z = \{z_i\}_{i=1}^r$, which is equipped with a binary operator $* : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$; here the log structure on $\mathbb{G}$ is given by

$$
\begin{aligned}
\mathbb{N}^r \oplus \mathbb{Z}_p[\![z]\!]^\times &\to \mathbb{Z}_p[\![z]\!] \\
(n, f) &\mapsto z^n f
\end{aligned}
$$

with the structure morphism $(\mathrm{id}, \mathrm{mod}\,(p, z)) : \mathbb{N}^r \oplus \mathbb{Z}_p[\![z]\!]^\times \to \mathbb{N}^r \oplus k^\times$, and $*$ is induced by the diagonal map on the level of rings and monoids.[5] The $*$ is associative and commutative. Moreover $\mathbb{G}$ is a group functor. Indeed let $\mathring{e}$ be the unique element in $\mathbb{G}(\mathring{\mathbb{Z}}_p)$. The pair $(*, \mathring{e})$ then turns $\mathbb{G}$ into a monoid with unit. Explicitly, for $(R_{(r)}, \mathfrak{m})$ an object of $\mathfrak{Art}$, we have

$$
\mathbb{G}(R_{(r)}, \mathfrak{m}) = \left\{ \begin{array}{ccc} \mathbb{N}^r \oplus \mathbb{Z}_p[\![z]\!]^\times & \to & \mathbb{N}^r \oplus R^\times \\ e_i & \mapsto & (e_i, \alpha_i) \end{array} \middle| \begin{array}{c} \alpha_i \equiv 1 \\ \mathrm{mod}\,\mathfrak{m} \end{array} \right\};
\tag{33}
$$

here the condition $\alpha \equiv 1 \mod \mathfrak{m}$ follows from the commutativity of the diagram

$$
\begin{array}{ccc}
(\mathbb{Z}_p[\![z]\!], \mathbb{N}^r \oplus \mathbb{Z}_p[\![z]\!]^\times) & \longrightarrow & R_{(r)} \\
& \searrow & \downarrow \\
& & k_{(r)}.
\end{array}
$$

We identify $\mathbb{G}(R_{(r)})$ with the set $(1 + \mathfrak{m})^r$ via the r-tuples (α_i) appeared in (33). Under this identification, $*$ corresponds the standard multiplication on $(1+\mathfrak{m})^r$. We remark that

---

[5] Notice however that $*$ is not representable.

the map $\mathrm{Spf}\,(\mathbb{Z}_p)_{(r)} \to \mathbb{G}$, whose log part reads $(\mathrm{id}, \mathrm{mod}\,(z)) : \mathbb{N}^r \oplus \mathbb{Z}_p[\![z]\!]^\times \to \mathbb{N}^r \oplus \mathbb{Z}_p^\times$, induces an isomorphism as functors on $\mathfrak{Art}$.

Let $\mathbb{T} = \mathrm{Spf}\,\mathbb{Z}_p[\![x-1]\!]$, $x = \{x_i\}_{i=r+1}^m$, be the formal torus over $\mathrm{Spf}\,\mathbb{Z}_p$ of dimension $(m-r)$, with characters generated freely by x. We regard $\mathbb{T}$ as a group functor on $\mathfrak{Art}$ by sending $R_{(r)}$ to $\mathbb{T}(R)$. The assignment

$$
\begin{aligned}
\mathrm{GL}_{m-r}(\mathbb{Z}_p) &\to \quad \{\text{basis of the characters of } \mathbb{T}\} \\
(\gamma_{ij})_{r<i,j\leq m} &\mapsto \quad \left\{ \prod_{j=r+1}^m x_j^{\gamma_{ij}} \right\}_{i=r+1}^m
\end{aligned}
\tag{34}
$$

establishes a bijection.

Let B be a $\mathbb{Z}_p[\![x-1]\!]$-algebra where

$$
B \cong \mathbb{Z}_p[\![x-1]\!] \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\![z]\!] \quad \text{non-canonically}
\tag{35}
$$

and is equipped with the log structure attached to the divisor $z = 0$. Let $\mathbb{S} = \mathrm{Spf}\,B$ regarded as a group functor formally sitting in the short exact sequence

$$
1 \longrightarrow \mathbb{G} \longrightarrow \mathbb{S} \longrightarrow \mathbb{T} \longrightarrow 1.
$$

Now fix a collection $\tilde{z} = \{\tilde{z}_i \in B\}_{i=1}^r$ of liftings of z_i. By an inspection of the group $\mathbb{S}((\mathbb{Z}_p)_{(r)})$, we see that different choices of the splitting (35) of B are in one-to-one correspondence with the changes of the collection $\tilde{z}$ in the way

$$
\tilde{z}_i \mapsto x^{\gamma_i} \tilde{z}_i, \quad \text{for some } \gamma_i \in (\mathbb{Z}_p)^{m-r}, 1 \leq i \leq r.
\tag{36}
$$

(b) The structure of S attached to a crystal

We will consider the following situation of an ordinary crystal with an explicitly described connection and Frobenius structure. In the following, we change the notations of the variables used in the main content slightly.

Let $\mathcal{H}$ be an ordinary Hodge F-crystal of weight one over S_0 with an A-basis $\{a_i, b_j\}$, where $(1,1) \leq (i,j) \leq (g,h)$ with $gh = m$, satisfying

$$
\mathrm{Fil}^1 \quad = \quad \text{the } A\text{-module generated by } \{b_j\}_{j=1}^h
\tag{37}
$$

$$
\nabla a_i = 0 \quad ; \quad \nabla b_j = \sum_{i=1}^g d\log q_{ji} \otimes a_i \quad \text{for some } q_{ji} \in A \text{ with } d\log q_{ji} \in \omega.
\tag{38}
$$

Suppose that there exists a subset E with r elements such that

$$
A = W[\![q_{ji} - \varepsilon_{ji}]\!] \quad ; \quad \mathcal{L} = \bigcup_{n \in \mathbb{N}^E} q^n A^\times
\tag{39}
$$

where $q = (q_{ji})$ and

$$
\varepsilon_{ji} = \begin{cases} 0 & \text{if } (i,j) \in E \\ 1 & \text{otherwise.} \end{cases}
$$

Denote by ϕ the lifting to S of the absolute Frobenius given by $\phi(q) = q^p$. Further suppose that the Frobenius structure of $\mathcal{H}$ is given by

$$\mathcal{F}(\phi)\phi^*a = a \quad ; \quad \mathcal{F}(\phi)\phi^*b_j = pb_j. \tag{40}$$

Now fix a labeling on the elements of E and its complement by $\{1, \cdots, r\}$ and $\{r + 1, \cdots, m\}$, respectively.

Proposition A.1 (Cf. [3, Th 2.1.14]) *Attached to the ordinary crystal $\mathcal{H}$ of weight one as above, one has a canonical isomorphism of log schemes*

$$S \cong \mathbb{S}, \tag{41}$$

where $\mathbb{S}$ is defined in (35).

Proof. With notations as above, the desired isomorphism $\mathbb{S} \to S$ is given by

$$q_l \mapsto \begin{cases} z_l & \text{if } l \leq r \\ x_i & \text{if } l > r. \end{cases}$$

Notice that different choices of the possible $\{a, b\}$ and q's for $\mathcal{H}$ satisfying the conditions (37) - (40) are exactly in one-to-one correspondence with the combinations of the transformations (34) and (36) for $\mathbb{S}$. Thus the isomorphism (41) is indeed canonical (cf. [3, Th 2.1.14]). $\qquad\square$

(c)   The periods of the $W_{(r)}$-points of S

With notations and assumptions as in the previous subsection, let $\overline{\mathcal{H}}_0 =$ the F-crystal over $k_{(r)}$ obtained by restriction of $\mathcal{H}$ to the origin (and forgetting the Hodge filtration). Then $\overline{\mathcal{H}}_0$ has the basis $\{\bar{a}, \bar{b}\} :=$ the restriction of $\{a, b\}$ to 0, with

$$\mathcal{F}\bar{a}_i = \bar{a}_i \quad ; \quad \mathcal{F}\bar{b}_j = p\bar{b}_j \quad ((1,1) \leq (i,j) \leq (g,h))$$
$$\mathcal{N}_{ls}\bar{a}_i = 0 \quad ; \quad \mathcal{N}_{ls}\bar{b}_j = \delta_{lj}\bar{a}_s \quad ((s,l) \in E).$$

Let e_ϕ be the Teichmüller lifting attached to ϕ and $\mathrm{Fil}_\phi^\bullet$ the Hodge filtration of the ordinary crystals $e_\phi^*\mathcal{H}$ over $k_{(r)}$. The $\bar{b}$ forms a basis of Fil_ϕ^1 over W.

Let $\chi \in S(W_{(r)})$ be a $W_{(r)}$-point of S and $\mathrm{Fil}_\chi^\bullet$ the Hodge filtration of $\chi^*\mathcal{H}$. Explicitly with the notations in the proof of Lemma 1.2, the element

$$\chi = (\chi^\flat, \chi^\sharp) : (A, \mathcal{L}) \to W_{(r)}$$

is determined by specifying $\alpha_{ji} \in pW$ and $\beta_{ji} \in 1 + pW$ such that

$$\chi^\flat(q_{ji} - 1) = \alpha_{ji} \quad \text{for } (i,j) \notin E$$
$$\chi^\sharp(e_{ji}, 1) = (e_{ji}, \beta_{ji}) \quad \text{for } (i,j) \in E.$$

In order to put all the q's into the same framework, we write

$$\beta_{ji} := \chi^\flat(q_{ji}) \in 1 + pW \quad \text{for } (i,j) \notin E.$$

Thus we have an isomorphism

$$
\begin{aligned}
S(W_{(r)}) &\cong (1+pW)^m \\
\chi &\mapsto (\beta_{ji}).
\end{aligned}
$$

Under this, the group structure, via the identification (41), of the left side coincides with the component-wise multiplication on the right side.

Proposition A.2 *With notations as above and under the canonical identification*

$$
\chi^*\mathcal{H} \cong e_\phi^*\mathcal{H} = \overline{\mathcal{H}}_0,
$$

given in (11), the W-module Fil_χ^1 (regarded as a submodule of $\overline{\mathcal{H}}_0$) is generated by the elements

$$
\bar{b}_j + \sum_{i=1}^{g} \varpi_{ji}\bar{a}_i \quad (1 \le j \le h)
$$

where $\varpi_{ji} = \log \beta_{ji} \in pW$.

Proof. The W-submodule Fil_χ^1 of $\chi^*\mathcal{H}$ is generated by χ^*b. Inductively on n, we obtain

$$
\begin{aligned}
\prod_{s=0}^{n-1}\left(\nabla\left(q_{ji}\frac{d}{dq_{ji}}\right) - s\right)b_l &= q_{ji}^n \nabla\left(\frac{d}{dq_{ji}}\right)^n b_l \\
&= (-1)^{n-1}(n-1)!\delta_{jl}a_i \quad (n > 0, 1 \le l \le h).
\end{aligned}
$$

Thus by the explicit formula (11) for the identification, the assertion follows. □

We call ϖ_{ji} the *periods* associated with χ. Let $\chi' \in S(W_{(r)})$ be another morphism corresponding to (β'_{ji}) with periods ϖ'_{ji} as above. Then clearly the sum $\chi + \chi' \in S(W_{(r)})$, via the group structure (41), has associated periods $\varpi_{ji} + \varpi'_{ji}$. One checks readily that the neutral element χ_0 is characterized by saying that the induced Hodge and the slope filtrations on $\chi_0^*\mathcal{H}$ coincide. This is the log analogue of [3, 2.1.3, 2.1.8].

Remark. The category $\mathfrak{Art}$ considered in this section does not contain the interesting log ring $W' := (W$ with the log structure attached to the divisor $\mathrm{Spec}\, k)$. It is tempting to find a better explanation of this exclusion or find a way to incorporate the direction "p" into the deformation. Notice however that the Frobenius on $k_{(1)}$ does not admit a lifting to W'.

References

[1] P. Berthelot, *Cohomologie cristalline des schémas de caractéristique $p > 0$*. Lecture Notes in Mathematics, Vol. 407. Springer-Verlag, Berlin-New York, 1974.

[2] P. Berthelot and A. Ogus, *Notes on crystalline cohomology*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[3] P. Deligne, Cristaux ordinaires et coordonnées canoniques. With the collaboration of L. Illusie. With an appendix by N. M. Katz. Lecture Notes in Math., 868, *Algebraic surfaces (Orsay, 1976–78)*, pp. 80-137, Springer, Berlin-New York, 1981.

[4] P. Deligne, Local behavior of Hodge structures at infinity. *Mirror symmetry, II*, 683-699, AMS/IP Stud. Adv. Math., 1, Amer. Math. Soc., Providence, RI, 1997.

[5] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces. *Comm. Math. Phys.* 167 (1995), no. 2, 301-350.

[6] O. Hyodo and K. Kato, Semi-stable reduction and crystalline cohomology with logarithmic poles. *Périodes p-adiques (Bures-sur-Yvette, 1988)*. Astérisque No. 223 (1994), 221-268.

[7] K. Kato, Logarithmic structures of Fontaine-Illusie. *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pp. 191-224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[8] N. M. Katz, Travaux de Dwork. *Séminaire Bourbaki, 24ème anné (1971/1972)*, Exp. No. 409, pp. 167-200. Lecture Notes in Math., Vol. 317, Springer, Berlin, 1973.

[9] M. Kontsevich, A. Schwarz and V. Vologodsky, Integrality of instanton numbers and p-adic B-model. *Phys. Lett. B* 637 (2006), no. 1-2, 97-101.

[10] C. Krattenthaler and T. Rivoal, On the integrality of the Taylor coefficients of mirror maps. *Duke Math. J.* 151 (2010), no. 2, 175-218.

[11] B. H. Lian and S.-T. Yau, Integrality of certain exponential series. *Algebra and geometry (Taipei, 1995)*, 215-227, Lect. Algebra Geom., 2, Int. Press, Cambridge, MA, 1998. (Also appeared as arXiv:hep-th/9507151.)

[12] A. Mokrane, La suite spectrale des poids en cohomologie de Hyodo-Kato. *Duke Math. J.* 72 (1993), no. 2, 301-337.

[13] A. Ogus, *F-crystals, Griffiths transversality, and the Hodge decomposition.* Astérisque No. 221 (1994).

[14] I. Shapiro, Frobenius map for quintic threefolds. *Int. Math. Res. Not. IMRN* 2009, no. 13, 2519-2545.

[15] J. Stienstra, Ordinary Calabi-Yau-3 crystals. *Calabi-Yau varieties and mirror symmetry (Toronto, ON, 2001)*, 255-271, Fields Inst. Commun., 38, Amer. Math. Soc., Providence, RI, 2003.

[16] V. Vologodsky, Integrality of instanton numbers. arXiv:0707.4617v3.

[17] J.-D. Yu, Notes on Calabi-Yau ordinary differential equations. *Commun. Number Theory Phys.* 3 (2009), no. 3, 475-493.