Notes on Calabi-Yau Ordinary Differential Equations*

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Abstract

We investigate the structures of Calabi-Yau differential equations and the relations to the arithmetic of the pencils of Calabi-Yau varieties behind the equations. This provides explanations of some observations and computations in the recent paper [14].

0 Introduction

This note may be regarded as a supplement to the paper [14] of Samol and van Straten. In that paper, the authors study the variation, along a projective line, of the Frobenius action on the cohomology of a Calabi-Yau variety (over a finite field) via the expansion at a totally degenerate point of a certain period of (a lifting of) the pencil. For certain families, they provide a p-adic analytic formula for the unit root as well as the Frobenius polynomial (in the case of rank 4) of each fiber provided the fiber is ordinary. Their explicit computations for those examples also reveal the existence of Dwork type congruences among the coefficients of the period of such a family.

In this note, we provide some theoretical explanations of the observations and computations in their paper. In particular, we show that certain cases of the Dwork congruences, including the mod p ones, follow from the relative geometry of the family.

This article is organized as follows. In §1, we explore the notion of ordinary differential equations of Calabi-Yau type over a field of characteristic 0 and derive some basic algebraic properties of them. The associated differential modules assemble the structures of the Gauss-Manin connection and the Poincaré pairing on certain parts of the relative de Rham cohomology groups of families of Calabi-Yau varieties over an affine line with totally degenerate fibers at the origin. In §2, we investigate some cohomological implications of the existence of a degenerate point of a pencil of Calabi-Yau varieties. For example, we see that the Galois representation on the cohomology of a certain degenerate fiber of a family over $\mathbb{Q}$ coincides with the representation attached to a modular form. §3 is devoted to the study of the mod p and p-adic properties of the Calabi-Yau equations. In particular, we study the relation between the solutions of a Calabi-Yau differential equation and the

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p-adic Hasse invariant of the underlying family. This relation provides an explanation of parts of the Dwork congruences. Unfortunately at this stage, we do not know how to deal with these Dwork congruences in general. Finally, some examples are given in §4, including a further inspection of the Hadamard products appeared in [14].

1 Ordinary differential equations of Calabi-Yau type

This section consists of exercises in the theory of differential modules. For basic references, see [20, Chap 2]. We recall the notion of differential equations of Calabi-Yau type and derive some basic properties of them. Such equations arise in the study of the Picard-Fuchs equations of pencils of Calabi-Yau varieties with certain degenerations. We postpone the geometric picture to the following sections. Notice that some properties we derived here (e.g., Lemma 1.3) can be obtained more easily if the differential equations are from geometry.

Let K be a field of characteristic 0 with a fixed embedding into $\mathbb{C}$. Denote by $\overline{K} \subset \mathbb{C}$ the algebraic closure of K. Let t denote a variable. It will also be regarded as a fixed parameter of the projective line $\mathbb{P}^1$ over K. Let $\theta = t\frac{d}{dt}$ be the usual logarithmic derivation with respect to t. In this paper, we adapt the point of view that the derivative θ near a regular singular point is more natural with respect to the logarithmic structure associated to the divisor $\{0\} \subset \mathbb{P}^1$. For this reason we will use the highly *non-standard* convention:

$$g' := \theta g$$

for a differentiable function $g = g(t)$ of t throughout the discussion. Now consider an ordinary linear differential operator of order $n \geq 1$ of the form

$$\mathcal{L} = \theta^n + \sum_{i=0}^{n-1} a_i \theta^i \tag{1}$$

with coefficients $a_i = a_i(t) \in K(t)$.

(a) The condition (N) and the β-factor

Suppose $\mathcal{L}$ is a differential operator of order n of the form (1). Consider the following condition on $\mathcal{L}$:

(N) Null exponents: $\mathcal{L}$ has a regular singularity at the origin $t = 0$ (i.e., $a_i \in K(t) \cap K[\![t]\!]$) and the associated indicial polynomial of $\mathcal{L}$ at this point reduces to

$$s^n + \sum_{i=0}^{n-1} a_i(0) s^i = s^n. \tag{2}$$

For any $\mathcal{L}$ as in (1), let formally

$$\beta := \exp\left(\frac{2}{n} \int a_{n-1} \frac{dt}{t}\right) \tag{3}$$

(i.e. a *non-zero* solution of $n\beta' = 2a_{n-1}\beta$). We call it the *β-factor* of $\mathcal{L}$. It exists at least in some differential field extension of $\overline{K}(t)$.

Lemma 1.1 *Suppose $\mathcal{L}$ as in (1) satisfies the condition (N). Then we have the following.*

(i) *The β-factor in (3) can be taken from $(1 + tK[\![t]\!])$.*

(ii) *Regard $\mathcal{L}$ as a differential operator in $\overline{K}(t)[\theta]$. Then the differential Galois group of $\mathcal{L}$ is contained in $\mathrm{SL}(n)_{\overline{K}}$ if and only if β can be chosen with*

$$\sqrt{\beta^n} \in K(t) \cap (1 + tK[\![t]\!])\,.$$

Proof. By (2), $a_{n-1}(0) = 0$ and hence one can formally choose

$$\int a_{n-1}\frac{dt}{t} \in tK[\![t]\!].$$

Thus the first assertion follows. The differential Galois group of $\mathcal{L} \in \overline{K}(t)[\theta]$ is in $\mathrm{SL}(n)$ if and only if there is a non-zero solution in $\overline{K}(t)$ of the operator $\theta + a_{n-1}$ ([20, Exercise 1.35.5]). Thus one can require that

$$\beta \in \sqrt[2/n]{\overline{K}(t)} \cap (1 + tK[\![t]\!])$$

and (ii) follows accordingly. □

From now on, we will always assume that $\beta \in (1 + tK[\![t]\!])$ if $\mathcal{L}$ satisfies the condition (N).

(b) The self-adjointness and the polarization

Let $\mathcal{L}$ be as in (1). Recall that the *formal adjoint* $\mathcal{L}^*$ of $\mathcal{L}$ is the differential operator

$$\mathcal{L}^* = (-1)^n\theta^n + \sum_{i=0}^{n-1}(-1)^i\theta^i a_i.$$

We say that $\mathcal{L}$ is *self-adjoint* if, as elements in $K(t)[\theta]$,

$$\mathcal{L}^* = (-1)^n\beta\mathcal{L}\beta^{-1}, \tag{4}$$

where β is defined in (3).

For any $\mathcal{L}$ of order n, denote by $\mathcal{M}_{\mathcal{L}}$ the left $K(t)[\theta]$-module with a generator η defined by:

$$\begin{array}{rcl} K(t) & \to & \mathcal{M}_{\mathcal{L}} := K(t)[\theta]/K(t)[\theta]\mathcal{L} \\ 1 & \mapsto & \eta, \end{array} \tag{5}$$

where the map is the natural projection. As a $K(t)$-module, $\mathcal{M}_{\mathcal{L}}$ is free of rank n with a basis $\{\eta^{(i)}\}_{i=0}^{n-1}$, where $\eta^{(i)} := \theta^i\eta$. An element $x \in \mathcal{M}_{\mathcal{L}}$ is called *horizontal* if $x' := \theta x = 0$.

Define a filtration $\mathrm{Fil}^\bullet$ on $\mathcal{M}_\mathcal{L}$ by setting Fil^i = the $K(t)$-submodule generated by $\{\eta^{(j)}\}_{j=0}^{n-1-i}$. A *polarization* on $\mathcal{M}_\mathcal{L}$ is a $K(t)$-linear, $(-1)^{n+1}$-symmetric, non-degenerate horizontal pairing

$$\langle \, , \rangle : \mathcal{M}_\mathcal{L} \times \mathcal{M}_\mathcal{L} \to K(t)$$

such that $\langle \mathrm{Fil}^i, \mathrm{Fil}^{n-i} \rangle = 0$ for $0 \le i \le (n-1)$. As usual, we say $\mathcal{M}_\mathcal{L}$ is *polarizable* if there exists a polarization on it; $\mathcal{M}_\mathcal{L}$ is called *polarized* if it is equipped with an underlying polarization. The aim of this subsection is to prove the following.

Theorem 1.2 *Let $\mathcal{L}$ be as in (1) and β be defined in (3). Then $\mathcal{L}$ is self-adjoint with $\beta \in K(t)$ if and only if $\mathcal{M}_\mathcal{L}$ is polarizable.*

We first prove the following.

Lemma 1.3 *Let $\langle \, , \rangle$ be a $K(t)$-linear horizontal pairing on $\mathcal{M}_\mathcal{L}$ such that $\langle \eta, \eta^{(i)} \rangle = 0$ for $0 \le i < (n-1)$. Let $\gamma = \langle \eta, \eta^{(n-1)} \rangle$. We have*

(i) *The pairing is uniquely determined by γ and $\gamma = c\beta^{-1}$ for some $c \in K$.*

(ii) *The pairing is $(-1)^{n+1}$-symmetric.*

(iii) *The pairing is a polarization if $\gamma \neq 0$.*

Proof. Since $\langle \, , \rangle$ is horizontal, the values $\langle \eta, \eta^{(i)} \rangle, 0 \le i \le m$ uniquely determine $\langle \eta^{(i)}, \eta^{(j)} \rangle$ for all $i + j \le m$ by a simple inductive procedure of taking derivatives. Thus in our case, the pairing is uniquely determined by γ. We have $\langle \mathrm{Fil}^i, \mathrm{Fil}^{n-i} \rangle = 0$ for $0 \le i \le (n-1)$ and the pairing is trivial if $\gamma = 0$.

Since $\langle \eta, \eta^{(n-2)} \rangle = 0$, we have

$$\begin{aligned} 0 &= \langle \eta, \eta^{(n-2)} \rangle' \\ &= \langle \eta', \eta^{(n-2)} \rangle + \langle \eta, \eta^{(n-1)} \rangle, \end{aligned}$$

which implies that $\langle \eta', \eta^{(n-2)} \rangle = -\gamma$. Similarly one finds, for $0 \le i \le n-1$, that

$$\langle \eta^{(i)}, \eta^{(n-1-i)} \rangle = (-1)^i \gamma$$

by induction. In particular, the pairing is non-degenerate if $\gamma \neq 0$.

Assume $n = 2l$ is even. Then

$$\begin{aligned} \gamma &= \langle \eta, \eta^{(n-1)} \rangle \\ &= -\langle \eta', \eta^{(n-2)} \rangle \\ &\vdots \\ &= (-1)^{l-1} \langle \eta^{(l-1)}, \eta^{(l)} \rangle. \end{aligned}$$

Taking derivatives of the l equations above, rewriting $\eta^{(n)}$ in terms of $\{\eta^{(i)}\}_{i=0}^{n-1}$ via $\mathcal{L}$ and summing them up, one gets

$$l\gamma' = -a_{n-1}\gamma.$$

Thus by definition, $\gamma = c\beta^{-1}$ for some $c \in K$. The computation is similar for n odd.

We now prove the parity of $\langle \, , \rangle$. We may assume that the pairing is non-degenerate. Let $\mathcal{N} = \mathrm{Hom}_{K(t)}(\mathcal{M}_{\mathcal{L}}, K(t))$ and consider the natural pairing

$$(\, ,) : \mathcal{N} \times \mathcal{M}_{\mathcal{L}} \to K(t)$$

given by $(f, m) = f(m)$. We equip $\mathcal{N}$ with the differential module structure such that $(\, ,)$ is horizontal. Then indeed, as a $K(t)[\theta]$-module, $\mathcal{N}$ is generated by ξ with

$$(\xi, \eta^{(i)}) = \begin{cases} 1 & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}$$

([20, Exercise 2.12.6]). Denote $\xi^{(i)} = \theta^i \xi$. By a similar computation as above, one deduces that

$$(\xi^{(i)}, \eta^{(j)}) = \begin{cases} (-1)^i & \text{if } i + j = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Now the non-degenerate paring $\langle \, , \rangle$ induces an isomorphism f between $\mathcal{M}_{\mathcal{L}}$ and $\mathcal{N}$ and it sends η to $\gamma\xi$. Thus under f, we can regard $\{\xi^{(i)}\}_{i=0}^{n-1}$ as another basis of $\mathcal{M}_{\mathcal{L}}$. Moreover, since $\mathcal{N}$ and $\mathcal{M}_{\mathcal{L}}$ are dual to each other (indeed, $\mathcal{N} = \mathcal{M}_{\mathcal{L}^*}$), after switching the roles of $\mathcal{M}_{\mathcal{L}}$ and $\mathcal{N}$, we see that

$$\langle \xi^{(i)}, \eta^{(j)} \rangle = (-1)^{n-1} \langle \eta^{(j)}, \xi^{(i)} \rangle.$$

This completes the proof of the parity of $\langle \, , \rangle$. □

Proof of Thm 1.2. Suppose $\mathcal{L}$ is self-adjoint and $\beta \in K(t)$. Then by the Lemma above, we can equip $\mathcal{M}_{\mathcal{L}}$ with the polarization determined by setting $\langle \eta, \eta^{(n-1)} \rangle = \beta^{-1}$.

On the other hand, suppose $\mathcal{M}_{\mathcal{L}}$ is polarized. Then $\beta \in K(t)$ by the Lemma above. Multiplying by a non-zero constant, we may assume that $\langle \eta, \eta^{(n-1)} \rangle = \beta^{-1}$. Then $\beta\eta$ is dual to $\eta^{(n-1)}$ with respect to the basis $\{\eta^{(i)}\}_{i=0}^{n-1}$. Thus $\mathcal{L}^*\beta\eta = 0$ ([20, Exercise 2.12.6]) and hence $\mathcal{L}^* = (-1)^n \beta \mathcal{L} \beta^{-1}$. □

(c) Calabi-Yau differential equations and the q-coordinate

Definition. A differential operator $\mathcal{L} \in K(t)[\theta]$ of the form (1) is called *Calabi-Yau* if $\mathcal{L}$ is self-adjoint and satisfies condition (N) in §1(a).

Remark. It might be better to call such an $\mathcal{L}$ as above *locally* or *quasi*-Calabi-Yau since in literature (e.g., in [1], [14]), there are some integral conditions on solutions of $\mathcal{L}$ (cf. Thm 3.3) and here we do not require that $\mathcal{L}$ is regular singular away from 0. However there seems to be no unified definition yet.

For any $\mathcal{L}$ as in (1) satisfying the condition (N), we set

$$\Lambda_{\mathcal{L}} = K[\![t]\!][\theta] / K[\![t]\!][\theta]\mathcal{L}. \tag{6}$$

It is a $K[\![t]\!]$-lattice in the completion $\mathcal{M}_{\mathcal{L}} \otimes_{K[t]} K[\![t]\!]$ of $\mathcal{M}_{\mathcal{L}}$. We abuse the notation by denoting $\mathrm{Fil}^i \subset \Lambda_{\mathcal{L}}$ the induced filtration from $\mathrm{Fil}^i \subset \mathcal{M}_{\mathcal{L}}$.

Suppose $\mathcal{L}$ is Calabi-Yau. Denote by $F(t) = F_0(t)$ the unique formal power series solution of $\mathcal{L}$ with constant term 1 near $t = 0$

$$\mathcal{L}F(t) = 0, \quad F(t) \in 1 + tK[\![t]\!]; \tag{7}$$

for $n \geq 2$, denote by $F_i(t), 1 \leq i \leq n-1$, the solutions with logarithmic pole of the form

$$\mathcal{L}F_i(t) = 0, \quad \sum_{r=0}^{i} (-1)^r F_{i-r}(t) \cdot \frac{\log^r t}{r!} \in tK[\![t]\!]. \tag{8}$$

Thus $\{F_i\}$ forms the Frobenius basis of solutions near $t = 0$ of $\mathcal{L}$. Let

$$wr_i = wr(F, F_1, \cdots, F_i) := \det\left(F_r^{(s)}\right)_{0 \leq r,s \leq i} \tag{9}$$

be the wronskians of $\{F_r\}_{r=0}^{i}$ and set

$$q = \exp\left(\frac{F_1}{F}\right) \in t + t^2 K[\![t]\!]. \tag{10}$$

Thus $K[\![t]\!] = K[\![q]\!]$. We call q the *q-coordinate* of $\mathcal{L}$.

Theorem 1.4 *Suppose $\mathcal{L}$ in (1) is Calabi-Yau. With the notations above, there exists a unique increasing filtration $U_\bullet$ of $K[\![t]\!][\theta]$-submodules of $\Lambda_{\mathcal{L}}$ such that, for all i,*

$$\Lambda_{\mathcal{L}} = U_i \oplus \mathrm{Fil}^{i+1} \tag{11}$$

and U_{i+1}/U_i are trivial $K[\![t]\!][\theta]$-modules. Moreover, up to a multiplicative constant, there exists a unique sequence $\{u_i \in \Lambda_{\mathcal{L}}\}_{i=0}^{n-1}$ with the following two properties:

(i) *As a $K[\![t]\!]$-module, U_i is generated by $\{u_r\}_{r=0}^{i}$.*

(ii) *u_i is of the form $u_i = \sum_{r=0}^{n-1-i} v_{i,r}\eta^{(r)}$ with*

$$v_{i,n-1-i} = \beta \frac{wr_i}{wr_{i-1}}. \tag{12}$$

Consequently, we have $u'_{i+1} = \tau_{i+1} \cdot u_i$, where

$$\tau_{i+1} = \frac{wr_{i-1} wr_{i+1}}{wr_i^2},$$

and in particular, $\tau_1 = (\log q)'$.

Proof. We find $\{u_i\}$ satisfying (11) and (12) by induction.

The condition (N) of $\mathcal{L}$ implies that, up to a constant multiple, there is a unique non-zero element

$$u_0 = \sum_{r=0}^{n-1} v_{0,r}\eta^{(r)}, \ v_{0,r} \in K[\![t]\!]$$

which is horizontal. Since

$$\mathcal{L}\langle u_0, \eta\rangle = \langle u_0, \mathcal{L}\eta\rangle = 0,$$

we see that $v_{0,n-1} = c\beta F$ for some constant c by Lemma 1.3. It is obvious that $c \neq 0$. Thus after modifying by a constant, $v_{0,n-1}$ is of the form (12).

Let $\mathcal{L}^{[i]}$ be the i-th exterior product of $\mathcal{L}$. Then $\mathcal{L}^{[i]}$ satisfies the condition (N) and $\Lambda_{\mathcal{L}^{[i]}}$ is a quotient of the i-th exterior power of the $K[\![t]\!]$-module $\Lambda_{\mathcal{L}}$. There exists a unique (up to a scalar) horizontal $u^{[i]} \in \Lambda_{\mathcal{L}^{[i]}}$ of the form

$$u^{[i]} = \sum_{r_1 \leq \cdots \leq r_i} v^{[i]}_{r_1 \cdots r_i} \eta^{(r_1)} \wedge \cdots \wedge \eta^{(r_i)}$$

with $v^{[i]}_{n-i,\cdots,n-1} \neq 0$. (Notice that $\eta^{(n-i)} \wedge \cdots \wedge \eta^{(n-1)} \neq 0$ in $\Lambda_{\mathcal{L}^{[i]}}$.) With the induced pairing,

$$\mathcal{L}^{[i]}\langle u^{[i]}, \eta \wedge \cdots \wedge \eta^{(i-1)}\rangle = \langle u^{[i]}, \mathcal{L}^{[i]}\left(\eta \wedge \cdots \wedge \eta^{(i-1)}\right)\rangle = 0.$$

Thus we have, after modifying by a scalar, $v^{[i]}_{n-i,\cdots,n-1} = \beta^i w r_i \in K[\![t]\!]^\times$. Therefore, by subtracting an element in U_{i-1}, we obtain u_i satisfying (11) and (12). $\square$

Corollary 1.5 *Suppose $\mathcal{L}$ in (1) is Calabi-Yau. With notations in Thm 1.4, we have $\langle U_{i-1}, U_{n-1-i}\rangle = 0$ and $\tau_i = \tau_{n-i}$ for all $1 \leq i \leq n$.*

Proof. By the last assertion of Thm 1.4, we see that $K(t) \otimes_{K[t]} U_i$ are the only possible $K(\!(t)\!)[\theta]$-submodules of $K(t) \otimes_{K[t]} \Lambda_{\mathcal{L}}$. Since U_i is stable under θ, so is its orthogonal complement $U_i^{\perp}$. Thus by a rank counting, $U_i^{\perp} = U_{n-2-i}$.

Since u_i lies in Fil^i and $\langle \mathrm{Fil}^i, \mathrm{Fil}^{n-i}\rangle = 0$, we have $\langle u_i, u_{n-1-j}\rangle = 0$ if $i \neq j$ and $\langle u_i, u_{n-1-i}\rangle$ are non-zero constants. Consequently,

$$\begin{aligned} 0 &= \langle u_i, u_{n-i}\rangle' \\ &= \tau_i \cdot \langle u_{i-1}, u_{n-i}\rangle + \tau_{n-i} \cdot \langle u_i, u_{n-1-i}\rangle. \end{aligned}$$

Since $\tau_i(0) = 1$ for all i by definition, the second assertion follows. $\square$

(d) Examples: lower order cases

In the remaining of this section, we consider Calabi-Yau $\mathcal{L}$ in the form (1) of lower orders more explicitly.

Suppose $n = 1$. Then condition (4) is empty. $\Lambda_{\mathcal{L}}$ is generated by $\beta F \eta$, which is horizontal.

Suppose $n = 2$. Then condition (4) is empty. $\Lambda_{\mathcal{L}}$ is generated by $\{u_0, u_1\}$ given by

$$\begin{aligned} u_0 &= \beta[F\eta' - F'\eta], \\ u_1 &= \frac{\eta}{F}. \end{aligned}$$

One computes easily that $u_0' = 0$ and $u_1' = (\log q)' u_0$. See §4(a) for a concrete example.

Suppose $n = 3$. We have the following.

Proposition 1.6 *A differential operator* $\mathcal{J} = \theta^3 + \sum_{i=0}^{2} b_i\theta^2 \in K(t)[\theta]$ *of order 3 is Calabi-Yau if and only if it is the symmetric square of a Calabi-Yau* $\mathcal{L}$ *in the form (1) of order 2.*

Proof. The operator $\mathcal{J}$ is the symmetric square of $\mathcal{L}$ if and only if

$$\begin{aligned} b_2 &= 3a_1 \\ b_1 &= 4a_0 + a_1' + 2a_1^2 \\ b_0 &= a_0' + 2a_0a_1. \end{aligned} \tag{13}$$

From the first two relations in (13), we see that the pair (b_2, b_1) determines the pair (a_1, a_0) uniquely. On the other hand, if we rewrite the last equation in (13) in terms of b_i, then it is equivalent to the condition (4) on $\mathcal{J}$. (Explicitly, the condition is equivalent to the relation

$$2\beta b_0 = (\beta b_1)' - (\beta b_2)'' + \beta^{(3)},$$

where β is the β-factor (3) of $\mathcal{J}$.) Finally if $\mathcal{J}$ is indeed the symmetric square of $\mathcal{L}$, it is easy to check that $\mathcal{J}$ satisfies (N) if and only if $\mathcal{L}$ does too. □

Corollary 1.7 *Let* $\mathcal{J} = \theta^3 + \sum_{i=0}^{2} b_i\theta^2 \in K(t)[\theta]$ *be Calabi-Yau of order 3. There exists a basis* $\{v_i\}_{i=0}^{2}$ *of* $\Lambda_{\mathcal{J}}$ *satisfying condition (11) and (12) in Thm 1.4 with*

$$v_2' = (\log \check{q})' v_1,$$

where $\check{q}$ *is the q-coordinate (10) of* $\mathcal{J}$*.*

Proof. By the lemma above, there is a Calabi-Yau $\mathcal{L}$ such that $\Lambda_{\mathcal{J}}$ is the symmetric square of $\Lambda_{\mathcal{L}}$ as $K[\![t]\!][\theta]$-modules. Let $\{u_0, u_1\}$ be the basis of $\Lambda_{\mathcal{L}}$ constructed in the discussion of the case $n = 2$ above. Let

$$v_0 = u_0^2, \ v_1 = u_0u_1, \ v_2 = \frac{1}{2}u_1^2.$$

Then they form a basis of $\Lambda_{\mathcal{J}}$ and satisfy the derivative conditions. □

Remark. In the case of the corollary above, $\check{q}$ coincides with the q-coordinate of the $\mathcal{L}$ in the proof.

(e) The case $n = 4$ *and* 5

Let us now consider the case when $n = 4$. Let $\mathcal{L}$ as in (1) be Calabi-Yau. Explicitly, the condition (4) translates to the relation on the coefficients of $\mathcal{L}$:

$$a_1 = a_2' + \frac{1}{2}a_2a_3 - \frac{1}{2}a_3'' - \frac{3}{4}a_3a_3' - \frac{1}{8}a_3^3. \tag{14}$$

Via the β-factor (3) of $\mathcal{L}$, equation (14) is equivalent to

$$\beta a_1 - (\beta a_2)' + (\beta a_3)'' - \beta''' = 0.$$

Proposition 1.8 *With the notation above, assume that $\mathcal{L}$ in (1) is a Calabi-Yau differential equation of order 4. Let $\beta, F, G := F_1$ be as given in (3), (7), (8), respectively. Consider the following elements in the $K[\![t]\!][\theta]$-module $\Lambda_{\mathcal{L}}$:*

$$\begin{aligned} u_0 &= \beta[F\eta''' - F'\eta'' + F''\eta' - F'''\eta] + \beta'[F\eta'' - F''\eta] + (\beta a_2 - \beta'')[F\eta' - F'\eta], \\ u_1 &= \frac{\beta}{F}\Big[(FG' - F'G)\eta'' - (FG'' - F''G)\eta' + (F'G'' - F''G')\eta\Big], \\ u_2 &= \frac{F\eta' - F'\eta}{FG' - F'G}, \\ u_3 &= \frac{\eta}{F}. \end{aligned}$$

Then $\{u_i\}_{i=0}^3$ forms a basis of $\Lambda_{\mathcal{L}}$ and satisfies conditions (11) and (12) in Thm 1.4. We have

$$\begin{aligned} u_2' &= \kappa \cdot (\log q)' \cdot u_1 \\ u_3' &= (\log q)' \cdot u_2, \end{aligned}$$

where q is the q-coordinate of $\mathcal{L}$ and

$$\kappa = \left(q\frac{d}{dq}\right)^2 \left(\frac{F_2}{F}\right) \in K[\![t]\!] = K[\![q]\!].$$

Proof. By a direct computation, we have $u_0' = 0$ and $u_3' = (\log q)' \cdot u_2$. Also one derives easily that

$$\begin{aligned} u_2' &= \frac{F^2}{\beta(FG' - F'G)^2} \cdot u_1 \\ &= \kappa \cdot (\log q)' \cdot u_1, \end{aligned}$$

where the second equality comes from [1, Prop 1]. Finally $u_1 = (\log q)' \cdot u_0$ by Thm 1.4. □

Remark. In terms of the α-factor introduced in [22, Lemma 4.1], we have (up to a constant multiple)

$$\alpha = t^3\beta.$$

We continue to assume that $\mathcal{L} \in K(t)[\theta]$ with leading coefficient 1 is Calabi-Yau of order 4. Let

$$\check{\mathcal{L}} = \theta^5 + \sum_{i=0}^{4} \check{a}_i\theta^i \in K(t)[\theta]$$

be the second exterior power of $\mathcal{L}$. Notice that by the self-adjointness of $\mathcal{L}$, the operator $\check{\mathcal{L}}$ is of order 5 (see [1, Prop 2 and 3]). Explicitly, we have

$$\begin{aligned}
\check{a}_4 &= \frac{5}{2}a_3 \\
\check{a}_3 &= 2a_2 + 2a_3' + \frac{7}{4}a_3^2 \\
\check{a}_2 &= -a_1 + 4a_2' + \frac{7}{2}a_2a_3 \\
\check{a}_1 &= -4a_0 + 2a_1' + a_2^2 + a_2'' + \frac{3}{2}a_1a_3 + \frac{3}{2}a_2'a_3 + \frac{1}{4}a_2a_3^2 \\
\check{a}_0 &= -2a_0' + a_1'' - 2a_0a_3 + a_1a_2 + \frac{3}{2}a_1'a_3 + \frac{1}{4}a_1a_3^2.
\end{aligned} \tag{15}$$

Proposition 1.9 *Keep the assumptions and notations as above. Then $\check{\mathcal{L}}$ is Calabi-Yau and there exists a basis $\{v_i\}_{i=0}^4$ of $\Lambda_{\check{\mathcal{L}}}$ satisfying condition (11) and (12) in Thm 1.4 with*

$$v_1' = \kappa \cdot (\log q)' \cdot v_0 \quad , \quad v_2' = (\log q)' \cdot v_1,$$
$$v_3' = (\log q)' \cdot v_2, \quad , \quad v_4' = \kappa \cdot (\log q)' \cdot v_3,$$

where q, κ are defined in Thm 1.8. Consequently the q-coordinate $\check{q}$ of $\check{\mathcal{L}}$ satisfies

$$d \log \check{q} = \kappa \cdot d \log q.$$

Proof. The self-adjointness is proved by a direct computation (see below for the explicit relations). Let $\{u_i\}_{i=0}^3$ be the basis of $\Lambda_{\mathcal{L}}$ constructed in Thm 1.8. Being the exterior power of $\Lambda_{\mathcal{L}}$, the $K[\![t]\!][\theta]$-module $\Lambda_{\check{\mathcal{L}}}$ is isomorphic to the quotient of $\bigwedge^2_{K[\![t]\!]} \Lambda_{\mathcal{L}}$ modulo the condition that $(u_0 \wedge u_3 - u_1 \wedge u_2)$ is horizontal. (cf. [1, Prop 2]). Put

$$v_0 = u_0 \wedge u_1,\ v_1 = u_0 \wedge u_2,\ v_2 = \frac{1}{2}(u_0 \wedge u_3 + u_1 \wedge u_2),\ v_3 = \frac{1}{2}u_1 \wedge u_3,\ v_4 = \frac{1}{2}u_2 \wedge u_3.$$

One then checks readily that they do the jobs.

On the other hand, with the notations in §1(c), we have

$$\begin{aligned}
\log \check{q} &= \frac{FF_2' - F'F_2}{FF_1' - F'F_1} \\
&= q\frac{d}{dq}\left(\frac{F_2}{F}\right) \\
&= \int \kappa \, d \log q.
\end{aligned}$$

Here we take the integral congruent to $\log q$ modulo $tK[\![t]\!]$. □

Conversely, let $\mathcal{J} = \theta^5 + \sum_{i=0}^4 b_i\theta^i \in K(t)[\theta]$ be a Calabi-Yau differential operator of order 5. If β is the β-factor of $\mathcal{J}$, the self-adjointness of $\mathcal{J}$ is equivalent to the following two relations:

$$\begin{aligned}
0 &= 2(\beta b_2) - 3(\beta b_3)' + 4(\beta b_4)'' - 5(\beta)^{(3)} \\
2(\beta b_0) &= (\beta b_1)' - (\beta b_2)'' + (\beta b_3)^{(3)} - (\beta b_4)^{(4)} + (\beta)^{(5)},
\end{aligned} \tag{16}$$

which together are equivalent to the two relations in [14, Prop 2.3].

Proposition 1.10 *Let $\mathcal{J}$ be Calabi-Yau of order 5 as above. Then there exists a unique Calabi-Yau $\mathcal{L} \in K(t)[\theta]$ of the form (1) of order 4 such that $\mathcal{J}$ is the second exterior power of $\mathcal{L}$.*

Proof. By the first four formulas in (15), we can determine the coefficients of $\mathcal{L}$ uniquely from those of $\mathcal{J}$. One checks that the condition (14) follows from (16). The condition (N) on $\mathcal{L}$ is obvious. $\square$

2 Degenerations

For the definitions and basic properties of logarithmic structures, see [7] or [5].

Fix a base field k. Consider a flat projective pencil $\pi : X \to \mathbb{P}^1$ whose generic fiber is smooth. We further assume that each singular fiber of π is a union of reduced divisors with normal crossings. (Over characteristic zero, this is possible by resolution of singularities and by passing to a finite steps of base change of cyclic covering from $\mathbb{P}^1$ to $\mathbb{P}^1$.) For such a pencil π, we equip X and $\mathbb{P}^1$ with the natural logarithmic structures associated to the union of the singular fibers (which is a reduced normal crossing divisor on X) and the critical values (which form a reduced divisor on $\mathbb{P}^1$), respectively. Then π is log-smooth. Denote by $\omega^i = \omega^i_{X/\mathbb{P}^1}$ the (locally free) sheaf on X of relative differential i-forms with log poles with respect to the log structures.

Now suppose that the generic fiber of π is an absolutely irreducible Calabi-Yau variety of dimension $m \geq 1$. We will call such a π a *nice pencil of Calabi-Yau varieties of dimension m*. Then the sheaf $\pi_*\omega^m$ is an invertible sheaf on $\mathbb{P}^1$. Suppose there exists a locally direct factor $\mathcal{M}$ of $R^m\pi_*\omega^\bullet$ of rank $(m+1)$ which contains $\pi_*\omega^m$ and is stable under the Gauss-Manin connection ∇. Now suppose $k = \mathbb{C}$. Let $a \in \mathbb{P}^1(\mathbb{C})$ be a $\mathbb{C}$-valued point and let N denote (the logarithm of) the local monodromy around a. Then N acts on the stalk $\mathcal{M}_a$ at a and is nilpotent.

We make the following working definition, which is a special variant of being Hodge-Tate in the sense of Deligne ([2, §6]).

Definition. With notations and assumptions as above, we call $\mathcal{M}$ *totally degenerate* at a if $N^m \neq 0$ on $\mathcal{M}_a$. It is called *of rigid type* at a if $N^{m-1} = 0$ but $N^{m-2} \neq 0$ on $\mathcal{M}_a$. We will abuse the notation by saying that the fiber at a of the family π is totally degenerate (resp. of rigid type) if there exists an $\mathcal{M}$ as above which is totally degenerate (resp. of rigid type) at a. We call $\mathcal{M}$ the *degenerate factor* of π in this case.

Remark. Suppose there is a totally degenerate fiber of $\pi : X \to \mathbb{P}^1$ over $\mathbb{C}$. Then the degenerate factor $\mathcal{M}$ is the unique irreducible locally direct factor of $R^m\pi_*\omega^\bullet$ which contains $\pi_*\omega^m$ and is stable under ∇.

For example, consider the case when $m = 3$. Let $\mathcal{M} \supset R^3\pi_*\omega^\bullet$ be of rank 4, which is locally a direct summand and is stable under ∇. Let N be the local monodromy around a point $a \in \mathbb{P}^1(\mathbb{C})$. Since $N^i : \mathrm{Gr}^W_{3+i}\,\mathcal{M}_a \to \mathrm{Gr}^W_{3-i}\,\mathcal{M}_a(-i)$ is an isomorphism (of Hodge structures) and the Hodge filtration Fil^i is locally free for each integer i, there are three possibilities of the degeneration types of the Hodge structure on $\mathcal{M}$ at this point:

(i) No degeneration ($N = 0$ on $\mathcal{M}$).

(ii) The fiber is of rigid type. In this case, $h^{1,1} = h^{3,0} = h^{0,3} = h^{2,2} = 1$. That is, the Hodge structure $\mathcal{M}_a$ is a consecutive extension of the Tate $\mathbb{C}(-2)$ of weight 4 by a rigid (= rank 2) Calabi-Yau piece of weight 3 by the Tate $\mathbb{C}(-1)$ of weight 2.

(iii) The fiber is totally degenerate. In this case, $h^{0,0} = h^{1,1} = h^{2,2} = h^{3,3} = 1$. That is, the Hodge structure $\mathcal{M}_a$ is a consecutive extension of Tate $\mathbb{C}(-i)$ of weights $2i = 6, 4, 2, 0$.

Lemma 2.1 *Let $\pi : X \to \mathbb{P}^1$ over $\mathbb{C}$ be a nice pencil of Calabi-Yau varieties of dimension m as above with a totally degenerate fiber at 0. Then the Poincaré pairing is non-degenerate around 0 on the degenerate factor $\mathcal{M}$.*

Proof. Let η be a local basis of sections of $\pi_*\omega^m$ near 0. By assumption, we have, for $0 \le i \le m$,

$$\eta^{(i)} \in \mathrm{Fil}^{m-i} \setminus \mathrm{Fil}^{m-i+1}$$

and they form a local basis of $\mathcal{M}$ at 0. Here $\eta^{(i)} = (\nabla(\theta))^i\eta$ ($= N^i\eta$ at 0). Since $(\mathrm{Fil}^i)^\perp = \mathrm{Fil}^{m+1-i}$, the cup-product $\gamma := \langle \eta, \eta^{(m)} \rangle$ is an invertible function near 0. By Lemma 1.3, the assertion follows. □

Corollary 2.2 *Let $\pi : X \to \mathbb{P}^1$ over $K \subset \mathbb{C}$ be a nice pencil of Calabi-Yau varieties of dimension m with a totally degenerate fiber at 0. Let η be a local basis of sections of $\pi_*\omega^m$ at 0 and let $\mathcal{L}$ be the Picard-Fuchs operator of η. Then $\mathcal{L}$ is a Calabi-Yau differential equation with respect to the parameter t of order $(m+1)$.*

Proof. The self-adjointness of $\mathcal{L}$ follows from Lemma 2.1. The validity of condition (N) is obvious by the total degeneracy assumption. □

Now suppose $k = \mathbb{Q}$. Assume that the nice pencil $\pi : X \to \mathbb{P}^1$ over $\mathbb{Q}$ of Calabi-Yau varieties of dimension m is of rigid type at $a \in \mathbb{P}^1(\mathbb{Q})$. Denote by $W_\bullet$ the corresponding monodromy filtration of the degenerate factor $\mathcal{M}_a$ at a. Then

$$\dim_{\mathbb{Q}} W_i/W_{i-1} = \begin{cases} 0 & \text{if } i \text{ odd and } i \neq m \\ 1 & \text{if } i \text{ even, } 2 \le i \le 2m-2 \text{ and } i \neq m \\ 2 & \text{if } i = m \text{ is odd} \\ 3 & \text{if } i = m \text{ is even.} \end{cases}$$

If $m \ge 3$, the subquotient $\mathcal{N} = W_m/(W_{m-1} + NW_{m+2})$ is then of rank two.

Proposition 2.3 *Fix a rational prime ℓ. With notations and assumptions as above, the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation on the ℓ-adic étale realization $\mathcal{N}_{et}$ of $\mathcal{N}$ coincides with the representation attached to a cusp form of weight $(m+1)$.*

Proof. One knows $\mathcal{N}_{et}$ is irreducible by [17, (4.8.9)]. If m is odd, the statement is a consequence of Serre's conjecture ([17, (3.2.4?) and Th 6]), which has recently been proved in [10]. If m is even, one uses [11, Cor 1.4]. □

3 Mod p and p-adic aspects

In this section, we fix a prime p and suppose $p > m$. Let (for simplicity) K be a finite unramified extension of $\mathbb{Q}_p$ with ring of integers W and residue field k. Let $\pi : X \to \mathbb{P}^1$ be a nice pencil of Calabi-Yau varieties of dimension m over K with a totally degenerate fiber at 0. Let ω^i be the sheaf on X of relative differential i-forms with log-poles along the logarithmic structure. We fix a basis η near 0 of sections of $\pi_*\omega^m \subset R^m\pi_*\omega^\bullet$ and let $\mathcal{L}$ be the Picard-Fuchs operator of η as before. We call that π has *nice* reduction if π has a flat model over W such that

(i) the reduction $\bar{\pi} : \bar{X} \to \mathbb{P}^1$ over k is also a nice pencil of Calabi-Yau varieties, and

(ii) the two log structures of π and $\bar{\pi}$ are induced from a smooth log structure on the flat model.

(a) The p-adic input

Lemma 3.1 *Suppose the pencil π has nice reduction and the degenerate factor $\mathcal{M}_0$ at 0 is stable under (a lifting of) the absolute Frobenius. Then the Frobenius action on $\mathcal{M}_0$ is ordinary and consequently $\mathcal{M}$ is generically ordinary.*

Proof. (Cf. [21, Thm 2.2].) Since $\mathcal{M}$ is totally degenerate at 0, the Hodge structure on $\mathcal{M}_0$ is a successive extension of rank 1 Hodge structures of pairwise different weights. Since the Frobenius respects the Hodge and the weight filtrations (cf. [12, Remarques 3.28]), the result follows. □

Lemma 3.2 *Suppose the pencil π over K with a totally degenerate fiber at 0 has nice reduction and the degenerate factor $\mathcal{M}_0$ at 0 is stable under Frobenius. Then there exists a non-zero constant $c \in W$ such that $c\beta F \in W[[t]]$.*

Proof. By Lemma 3.1 and [8, Prop 3.1.3] (cf. [22, Cor 2.2]), there is a non-zero element

$$u = \sum_{i=0}^{m} v_i \eta^{(i)}, \ v_i \in W[[t]]$$

which is horizontal. By Thm 1.4, $v_m = c\beta F$ for some constant c. □

Theorem 3.3 *Let π be a nice pencil of Calabi-Yau varieties of dimension m over $\mathbb{Q}$. Fix a basis near 0 of sections of $\pi_*\omega^m$ and let $\mathcal{L}$ be the associated Picard-Fuchs operator. Let F be the formal solution of $\mathcal{L}$ as in (7). Then $F(t) \in (1 + t\mathbb{Z}_p[[t]])$ for all p sufficiently large. In particular, $F(t) \in \mathbb{Z}[1/N][[t]]$ for some integer $N \neq 0$.*

Proof. (Cf. [14, Conj 2.1].) By the Lemma above and its proof, we see that except for a finite number of primes, $\beta F \in \mathbb{Z}_p[[t]]$ for all the remaining p such that the pencil π has nice reduction. By Lemma 1.3, $\beta(t)$ is rational over $\mathbb{Q}$ and hence is in $(1 + t\mathbb{Z}_p[[t]])$ for almost all p. Thus the assertion follows. □

(b) The Hasse invariant and the differential equation

Assume that π has nice reduction $\bar{\pi}$. Let $\bar{\omega}^i$ be the sheaf on $\bar{X}$ of relative differential i-forms with log-poles. We choose η that can be extended to the flat model of π and assume the following condition is satisfied:

(R) Non-degeneracy of the reduction of η at the origin:

$$\langle \eta, \eta^{(m)} \rangle \in W^{\times} + tW[\![t]\!].$$

Let σ be the lifting to $W[\![t]\!]$ of the absolute Frobenius defined by sending t to t^p. For any $x \in W[\![t]\!]$, denote by x^σ the image under σ.

Consider the adjoint morphism

$$V : \bar{\pi}_* \bar{\omega}^m \to \sigma^* \bar{\pi}_* \bar{\omega}^m$$

of the absolute Frobenius with respect to the cup-product on $R^m \pi_* \bar{\omega}^\bullet$. Notice that V is horizontal since the Frobenius and the cup-product are. Represent V by $\mathfrak{H}$ defined by

$$V(\eta) = \mathfrak{H} \cdot \sigma^* \eta. \tag{17}$$

With our choice of the parameter, $\mathfrak{H}$ is an element in $k(t)$. We call $\mathfrak{H}$ the *Hasse invariant* of the family $\bar{\pi}$ (with respect to η). Notice that $\mathfrak{H}(0)$ is well-defined and non-zero at $t = 0$ by Lemma 3.1.

Proposition 3.4 *We have $\mathcal{L}\mathfrak{H} = 0$. That is, with respect to the parameter t, the Hasse invariant $\mathfrak{H}$ is a rational solution of $\mathcal{L}$ in characteristic p.*

Proof. By the very definition, $\nabla(\mathcal{L})\eta = 0$ and $\nabla \sigma^* \eta = 0$. Since V is horizontal, we see that $\mathcal{L}\mathfrak{H} = 0$ by applying $\nabla(\mathcal{L})$ to the equation (17). □

Thus, for example, if $\pi_* \bar{\omega}^m = \mathcal{O}(1)$ on $\mathbb{P}^1$, then $\mathfrak{H}$ is a section of $\mathcal{O}(p-1)$. Consequently, if $\eta \in \Gamma(\mathbb{P}^1, \pi_* \bar{\omega}^m)$ and regarding $\mathfrak{H}$ as a function of t, we see that $\mathfrak{H} \in k[t]$ is of degree $\leq (p-1)$.

If (R) is satisfied, then by Lemma 1.3, 3.2 and their proofs, $F \in W[\![t]\!]$. Let

$$\mathcal{H}(t) = \frac{F(t)}{F(t)^\sigma} \tag{18}$$

regarded as a formal power series over W. Let $\bar{\mathcal{H}}(t)$ be the reduction mod p of $\mathcal{H}(t)$.

Proposition 3.5 *Suppose π satisfies the condition (R) above. Then we have the following.*

(i) $\bar{\mathcal{H}}(t) \in (1 + tk[\![t]\!]) \cap k(t)$.

(ii) *The function $\bar{\mathcal{H}}$ satisfies $\mathcal{L}\bar{\mathcal{H}} = 0$.*

Proof. With the notations in Thm 1.4, the element $u_0 = \beta F \eta^{(m)} + \cdots$ is a local generator of the submodule U_0 of horizontal sections of the degenerate factor $\mathcal{M}$ (cf. the proof of Lemma 3.2). Then U_0 is a unit-root F-(iso)crystal ([8, Prop 3.1.3]) and hence by [8, 4.1.9], the function $\beta F/(\beta F)^\sigma$ is a lifting of an element in $k(t)$. Thus the first assertion follows.

Formally we have

$$F(t) = \frac{F(t)}{F(t)^\sigma} \frac{F(t)^\sigma}{F(t)^{\sigma^2}} \cdots = \mathcal{H}(t)\mathcal{H}(t)^\sigma \cdots .$$

Applying the differential operator $\mathcal{L}$, we have

$$\begin{aligned} 0 = \mathcal{L}F(t) &= \mathcal{L}\big(\mathcal{H}(t)\mathcal{H}(t)^\sigma \cdots\big) \\ &\equiv \big(\mathcal{L}\mathcal{H}(t)\big)\big(\mathcal{H}(t)^\sigma \cdots\big) \\ &\equiv \big(\mathcal{L}\bar{\mathcal{H}}(t)\big)\big(\bar{\mathcal{H}}(t)^p \cdots\big) \pmod p. \end{aligned}$$

Thus the rational function $\bar{\mathcal{H}}(t)$ is a solution of $\mathcal{L}$ in characteristic p. □

Proposition 3.6 *Assume condition (R) is fulfilled. Let $c = \mathfrak{H}(0)$. We have $\mathfrak{H} = c\bar{\mathcal{H}}$ regarded as rational functions of t.*

Proof. Over a non-empty open subset of $\mathbb{P}^1$, both functions $\mathfrak{H}$ and $c\bar{\mathcal{H}}$ represent the absolute Frobenius action on η ([8, 4.1.9]; cf. the proof of Prop 3.5(i)). Thus the assertion follows. □

Corollary 3.7 *Assume that condition (R) is fulfilled and $\pi_*\omega^m = \mathcal{O}(1)$. Suppose $\eta \in \Gamma(\mathbb{P}^1, \pi_*\bar{\omega}^m)$ and let $c = \mathfrak{H}(0)$. Then $c\bar{\mathcal{H}}(t) = \mathfrak{H}(t) \equiv cF^{<p}(t) \mod p$, where $F^{<p}(t)$ is the truncation of $F(t)$ up to degree $(p-1)$.*

Proof. Under the assumptions, $\mathfrak{H}(t)$ is a polynomial of degree $< p$. □

Remark. The statement of the corollary is equivalent to the mod p case of the Dwork congruences of the coefficients $\chi(n)$ of $F(t) = \sum \chi(n)t^n$ in [14, §2.3]. Indeed let $\nu = \sum_{i\geq 0} \nu_i p^i, 0 \leq \nu_i < p$, be the p-adic expansion of an integer $\nu \geq 0$. Then Cor 3.7 implies

$$\chi(\nu) \equiv \prod_{i\geq 0} \chi(\nu_i) \mod p,$$

which is equivalent to the mod p case.

On the other hand, if $\pi_*\omega^m = \mathcal{O}(n)$ for some $n > 1$, then $\bar{\mathcal{H}}$ is of higher degree. This may be used to detect the degree of $\pi_*\omega^m$ from the period F.

(c) The higher congruences

We keep the assumptions and notations in Lemma 3.1; assume that π has nice reduction $\bar{\pi}$ and (R) is fulfilled.

In this subsection, we are interested in understanding the higher Dwork congruences geometrically. For this purpose, we suppose that $\pi_*\omega^m = \mathcal{O}(1)$ over W and $\eta \in \pi_*\omega^m(\mathbb{P}^1)$ is globally defined. Thus β^{-1}, which equals $\langle \eta, \eta^{(m)} \rangle$ up to a multiplicative constant, is in $W[t]$.

We use the following notations. Let $\mathcal{H}_0 = F^{<p}(t)$. Denote by $\Delta_0 = \operatorname{Spec} k[t]/\bar{\mathcal{H}}$ the *Hasse locus* of the pencil $\bar{\pi}$. Let $\Delta_\infty = (\mathbb{A}^1_W)_{/\Delta_0}$ be the completion of the affine line $\mathbb{A}^1_W$ along Δ_0. Define

$$R_\infty = \varprojlim_n W[t, \mathcal{H}_0^{-1}]/(p^{n+1}),$$

and let

$$S_\infty = \operatorname{Spf} R_\infty.$$

Under this circumstance, $\mathcal{H} \in R_\infty$ is a section of $\mathcal{O}(p-1)$ with $\mathcal{H} \equiv \mathcal{H}_0 \mod p$ (cf. Cor 3.7 and [8, 4.1.9]; locally $\mathcal{H}$ defined in (18) represents the absolute Frobenius on $\beta^{-1}u$ up to a multiplicative constant, where u occurs in the proof of Lemma 3.2). On the other hand, the unit-root part of $\mathcal{M}$ defines a (multiplicative) formal group over S_∞.

We make a further assumption:

(E) The above formal group over S_∞ can be extended to a formal group G over the completion $(\mathbb{P}^1_W)_{/\mathbb{P}^1_k}$.

For example ([21]), the Dwork pencil has certain degenerate fibers; however the attached formal group can be extended to the whole $(\mathbb{P}^1_W)_{/\mathbb{P}^1_k}$ in an explicit way by writing down its logarithm (see [21, Prop 5.2]). See [22, §3], [23, §4] for more examples.

Now suppose (E) holds. Represent G by a formal group law over the affine part $(\mathbb{A}^1_W)_{/\mathbb{A}^1_k}$ with logarithm

$$l(\tau) = \sum_{n=1}^{\infty} \alpha(n) \frac{\tau^n}{n}, \quad \alpha(n) \in \varprojlim W[t]/(p^{i+1}),$$

which is normalized by $\alpha(p^n) \equiv \mathcal{H}\alpha(p^{n-1})^\sigma \mod p^n$ for all $n \geq 1$. Write $F(t) = \sum \chi(n) t^n$. For any $1 \leq m < p$ and $s \geq 0$, consider the truncation of $F(t)$

$$F_{(mp^s)} = \sum_{n=(m-1)p^s}^{mp^s - 1} \chi(n) t^n.$$

Lemma 3.8 *With notations as above, let $\{a_n\}_{n=0}^\infty, a_n \in W[t]$, be a sequence with $a_0 \in W^\times$ and $a_{n+1} \equiv \mathcal{H} a_n^\sigma \mod p^{n+1}$ for all $n \geq 0$. Then we have the following.*

(i) *For any $b \in W[t]$ with $b \equiv a_1 \mod p$, there exists a sequence $\{b_n\}_{n=1}^\infty, b_n \in W[t]$, such that $b_1 = b$ and $b_{n+1} \equiv \mathcal{H} b_n^\sigma \mod p^{n+1}$ for all $n \geq 1$.*

(ii) *Fix a positive integer n. For any $c \in W[t]$ with $c \equiv a_n \mod p^n$, there exists an element $\tilde{c} \in W[t]$, unique modulo p^{n+1}, such that*

$$a_{n+1} \cdot c^\sigma \equiv a_n^\sigma \cdot \tilde{c} \mod p^{n+1}.$$

Proof. (i) Indeed, if $b = a_1 + p\delta$, one checks that the sequence of elements

$$b_n = a_n + p \cdot a_{n-1}\delta^{\sigma^{n-1}}, \quad n \geq 2$$

(with $b_1 = b$) satisfies the requirement.

(ii) The assumption implies $a_{n+1} \equiv a_1^{1+p+\cdots+p^n} \mod p$. Write $c = a_n + p^n\alpha$. We are asked to solve ε in

$$a_{n+1}\big(a_n + p^n\alpha\big)^\sigma \equiv a_n^\sigma\,(a_{n+1} + p^n\varepsilon) \mod p^{n+1},$$

which is equivalent to solve

$$a_n^\sigma\varepsilon \equiv a_{n+1}\alpha^\sigma \mod p.$$

The latter has a unique solution $a_1 \cdot \alpha^\sigma$ modulo p. □

Theorem 3.9 *With notations as above and under the assumption (E), the formal power series*

$$\tilde{l}(\tau) = \sum_{1\leq m<p}^{s\geq 0} F_{(mp^s)}\frac{\tau^{mp^s}}{mp^s}$$

is the logarithm of a formal group law $\tilde{G}$ over $(\mathbb{A}^1_W)_{/\mathbb{A}^1_k}$. Moreover, $\tilde{G}$ is strictly isomorphic to G defined above over S_∞.

Proof. We shall show that the sequence $\{F_{(mp^s)}\}$ satisfies the congruence property

$$F_{(mp^s)} \equiv \mathcal{H} \cdot F^\sigma_{(mp^{s-1})} \mod p^s. \tag{19}$$

Consequently the theorem follows by [15, Thm (A.8), (A.9)].

For $s = 1$, the statement is a consequence of Cor 3.7.

Away from Δ_∞, we have

$$\alpha(m,s) := \frac{\alpha(mp^s)}{\alpha(mp^{s-1})^\sigma} \in R_\infty$$

and $\alpha(m,s) \equiv \mathcal{H} \mod p^s$ for all $s \geq 1$. To shorten the notation, first assume that $\mathcal{H} \mod p$ is exactly of degree $(p-1)$. Now since $\alpha(m,s)$ represents the absolute Frobenius modulo p^s, which is a section of the sheaf $\mathcal{O}(p-1)$ of $\mathbb{P}^1$ over the affine open $\operatorname{Spec} W[t, \mathcal{H}_0^{-1}]/(p^s)$, we have

$$\alpha(m,s) \equiv \frac{A}{\mathcal{H}_0^n} \mod p^s \tag{20}$$

for some n and $A \in W[t]/(p^s)$ of degree $(n+1)(p-1)$.

On the other hand, applying Lemma 3.8(i) with $a_n = \alpha_{1,n}$ and $b = F_{(mp)}/F_{(m)}$ ($\in W[t]$ with invertible constant term), we obtain a sequence $\{b_n\}$ such that, for $s \geq 1$,

$$\alpha(m,s) \equiv \frac{b_s}{b_{s-1}^\sigma} \equiv \mathcal{H} \mod p^s. \tag{21}$$

(Here we set $b_0 = F_{(m)}$.) By the first congruence in (21) together with (20), we see that $\deg b_s = mp^s - 1$. Now by a simple induction argument, the second congruence in (21) together with Lemma 3.8(ii) (for $\{b_n\}$) imply that one can take $b_s = F_{(mp^s)}$ for all $s \geq 0$.

In general, we represent the occurred finite degree elements as homogeneous elements in $W[t_0, t_1]$ (e.g., $t_0^{mp^s-1}\alpha(t_1/t_0)$ instead of α) to obtain that $\deg b_s < mp^s$ and then repeat the argument above. This completes the proof. $\square$

Corollary 3.10 *Write $F(t) = \sum \chi(n)t^n$. For all non-negative integers ν, m, s with $0 \leq \nu < p$, we have*

$$\chi(\nu + mp^{s+1}) \equiv \chi(\nu)\chi(mp^s) \mod p^{s+1}.$$

Proof. One checks readily that it suffices to establish the congruences for $1 \leq m < p$. Assuming m is in this range. The theorem above (see (19)) implies that, for all $s \geq 0$,

$$F \cdot F^{\sigma}_{(mp^s)} \equiv F_{(mp^{s+1})} \cdot F^{\sigma} \mod p^{s+1}. \qquad (\kappa_{m,s})$$

For $m = 1$, by comparing the coefficients of $t^{\nu+p^{s+1}}$ in both sides of the equation $(\kappa_{1,s})$, we get

$$\sum_{j=0}^{p^s-1} \chi(\nu + p^{s+1} - pj)\chi(j) \equiv \sum_{j=0}^{p^s-1} \chi(\nu + pj)\chi(p^s - j) \mod p^{s+1}.$$

By canceling common terms on the two sides of the congruence, the assertion (corresponding to $j = 0$) follows immediately. For $m > 1$, by comparing the coefficients of $t^{\nu+mp^{s+1}}$ in both sides of equations $(\kappa_{m,s})$ and $(\kappa_{m-1,s})$, one obtains the desired congruences by induction on m. $\square$

Question. Is there a geometric/homological interpretation of the general Dwork congruences or the supercongruent phenomena (see [13] and the references therein)?

(d) The higher Hasse invariant

Suppose that $\mathcal{L}$ satisfies the condition (R) and for simplicity that $R^m\pi_*\omega^\bullet$ is of rank $(m+1)$. Let $\mathcal{M}_{cris}$ be the relative m-th logarithmic crystalline cohomology of $\bar{\pi}$. Assume p odd. Consider $\mathcal{N} := \bigwedge^2 \mathcal{M}_{cris}$ equipped with the induced cup-product pairing and with Frobenius $= p^{-1} \cdot$ (the induced Frobenius from $\mathcal{M}_{cris}$ to $\bigwedge^2 \mathcal{M}_{cris}$). Then $\xi = \eta \wedge \eta'$ is a local section of $\mathcal{N}$ and $\not\equiv 0 \mod p$.

Let $\bar{\mathcal{N}} = (\mathcal{N} \mod p)$ and V the adjoint of the Frobenius on $\bar{\mathcal{N}}$. Define the Hasse invariant $\check{\mathfrak{H}}$ (with respect to ξ) of $\mathcal{N}$ by

$$\begin{aligned} V : \bar{\mathcal{N}} &\rightarrow \sigma^*\bar{\mathcal{N}} \\ \xi &\mapsto \check{\mathfrak{H}} \cdot \sigma^*\xi. \end{aligned}$$

If η is a section over an open subset U of $\mathbb{P}^1$, then we have

$$\check{\mathfrak{H}} \in \Gamma\left(U, \left(\pi_*\bar{\omega}^m \otimes R^1\pi_*\bar{\omega}^{m-1}\right)\right),$$

and for $x \in U(\bar{k})$, the Newton polygon of $\mathcal{M}_{cris}$ over x starts with slopes

$$\begin{cases} 0,1 & \text{if } \mathfrak{H}(x) \neq 0 \text{ and } \check{\mathfrak{H}}(x) \neq 0 \\ 0,>1 & \text{if } \mathfrak{H}(x) \neq 0 \text{ but } \check{\mathfrak{H}}(x) = 0 \\ 1/2, 1/2 & \text{if } \mathfrak{H}(x) = 0 \text{ but } \check{\mathfrak{H}}(x) \neq 0 \\ > 1/2 & \text{if } \mathfrak{H}(x) = 0 \text{ and } \check{\mathfrak{H}}(x) = 0. \end{cases}$$

Consequently the variation of crystals $\mathcal{M}_{cris}$ (over U) away from $\check{\mathfrak{H}} = 0$ is an extension ([9, Thm 2.4.2])

$$0 \to \mathcal{M}_{\leq 1} \to \mathcal{M}_{cris}\big|_{U\setminus\{\check{\mathfrak{H}}=0\}} \to \mathcal{M}_{>1} \to 0,$$

where $\mathcal{M}_{\leq 1}$ (resp. $\mathcal{M}_{>1}$) is the slope ≤ 1 (resp. > 1) part which is of rank 2 (resp. $m-1$).

On the other hand, let

$$\check{F}(t) = (FG' - F'G)(t) \in K[\![t]\!].$$

Then the condition (R) on $\mathcal{L}$ implies that indeed, $\check{F}(t) \in W[\![t]\!]$. Similarly to the discussion in the previous subsection, if $\pi_*\omega^m \otimes R^1\pi_*\omega^{m-1} = \mathcal{O}(1)$ and ξ is a global section, we have $\check{\mathfrak{H}}(t) \equiv \check{c}\check{F}^{<p}(t) \mod p$, where $\check{c} = \check{\mathfrak{H}}(0)$.

4 Examples

(a) The Legendre family

Let λ be a fixed parameter of $\mathbb{P}^1$. Consider the Legendre family $\pi : E \to \mathbb{P}^1$ of elliptic curves over $\mathbb{Q}$ whose affine part is given by

$$y^2 = x(x-1)(x-\lambda).$$

The Picard-Fuchs operator $\mathcal{L}$ associated to the invariant differential

$$\eta = \frac{dx}{2y}$$

is the one associated to the Gauss hypergeometric series

$$F(\lambda) := {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; \lambda).$$

Note that the monodromy around $\lambda = \infty$ is not unipotent. Consider the double cover $a : \mathbb{P}^1 \to \mathbb{P}^1$ given by $\lambda = t^{-2}$. Then

$$2\eta = \frac{dx}{\sqrt{x(x-1)(x-1/t^2)}} = \frac{tdx}{\sqrt{x(x-1)(t^2x-1)}},$$

which is zero precisely when $t = 0$. Thus with respect to t, we have $(a^*\pi)_*\omega^1_{a^*E/\mathbb{P}^1} = \mathcal{O}(1)$.

Let p be an odd prime. The discussion above shows that the Hasse invariant of the family over $\mathbb{F}_p$ in the affine part with respect to λ is a polynomial of degree $\leq \frac{p-1}{2}$. In fact,

$$\mathfrak{H}(\lambda) \equiv (-1)^{(p-1)/2} F^{<p}(\lambda) \mod p$$

is of degree exactly $\frac{p-1}{2}$. This is due to the fact that $\mathfrak{H}(\lambda)$ has only simple roots ([18, Thm V.4.1]) and that the singular curve corresponding to $t = 0$ is ordinary.

Let an upper $'$ denote the derivative with respect to $\log \lambda$. The set

$$\left\{u_0 = (1-\lambda)[F\eta' - F'\eta], u_1 = \frac{\eta}{F}\right\}$$

provides a local basis of

$$\mathcal{H} := R^1\pi_*\left(\omega^\bullet_{E|_{\mathbb{A}^1}/\mathbb{A}^1}\right)$$

near $\lambda = 0$ adapted to the slope filtration (cf. [8, §8]).

The Galois representation on the cohomology $\mathcal{H}_0$ over $\mathbb{Q}$ at $\lambda = 0$ is reducible. It has the form

$$0 \to \mathbb{Q}(\epsilon) \to \mathcal{H}_0 \to \mathbb{Q}(-1)(\epsilon) \to 0,$$

where ϵ is the Legendre character. This is simply because the corresponding singular curve splits over $\mathbb{Q}(\sqrt{-1})$ but not over $\mathbb{Q}$. This gives an explanation of the constant term ± 1 of $\mathfrak{H}(\lambda)$.

(b) Dwork families

Fix an integer $n \geq 2$. The Dwork family of Calabi-Yau varieties of dimension $(n-1)$ is the pencil of hypersurfaces in $\mathbb{P}^n$ given by the equation

$$\mathcal{P}_t : X_1^{n+1} + \cdots + X_{n+1}^{n+1} - (n+1)tX_1 \cdots X_{n+1}.$$

In this case, we consider the differential

$$\eta = \operatorname{Res} \frac{t\Omega}{\mathcal{P}_t},$$

where

$$\Omega = \sum (-1)^i X_i dX_1 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_{n+1}.$$

Via the parameter $\lambda = t^{-(n+1)}$, the associated Picard-Fuchs operator $\mathcal{L}$ is the one associated to the generalized hypergeometric series

$${}_nF_{n-1}\left(\begin{array}{c} \frac{1}{n+1}, \cdots, \frac{n}{n+1} \\ 1, \cdots, 1 \end{array}; \lambda\right).$$

Explicitly one can pick the annihilator of η to be

$$\begin{aligned} \mathcal{L} &= \theta_\lambda^n - \lambda \prod_{i=1}^n \left(\theta_\lambda + \frac{i}{n+1}\right) \\ &= (1-\lambda)\theta_\lambda^n - \frac{n}{2}\lambda\theta_\lambda^{n-1} + \cdots \end{aligned}$$

where $\theta_\lambda = \lambda\frac{d}{d\lambda}$. One checks that $\beta = (1-\lambda)$ is the β-factor of $\mathcal{L}$ regarded as a Calabi-Yau differential operator in $\mathbb{Q}(\lambda)[\theta_\lambda]$.

Similar to the case of the Legendre family, the monodromy at $\lambda = \infty$ is not unipotent. However with respect to t, the form η is well-defined everywhere and vanishes precisely when $t = 0$. Thus $\pi_* \omega^{n-1} = \mathcal{O}(1)$. Consequently for the reduction over $\mathbb{F}_p$, $p \nmid (n+1)$, the Hasse invariant $\mathfrak{H}(\lambda)$, regarded as a polynomial of λ here, is a polynomial of degree at most $\lfloor \frac{p-1}{n+1} \rfloor$. In fact the degree of $\mathfrak{H}(\lambda)$ is exactly the upper bound. This is because the Fermat point $t = 0$ is not ordinary if and only if $p \not\equiv 1 \mod (n+1)$ and there is no n-multiple root of "$\mathfrak{H}(t)$". Notice that in this example, double roots do occur in $\mathfrak{H}(\lambda)$.

On the other hand, over each geometric point of $\mu_{n+1} := \operatorname{Spec} \mathbb{Q}[t]/(t^{n+1} - 1)$ in $\mathbb{P}^1$, the fiber of the Dwork family has ordinary double points as its singularities. Thus $N^2 = 0$ for the local monodromy N around a point of μ_{n+1}. One can show that $N \neq 0$ in this case ([4, Cor 1.7]). Now consider the fiber over $t = 1$ in the case $n = 4$. Retain the notations in the end of §2. Then the fiber is of rigid type and the subquotient W_3/W_2 is modular, which was first proved by Schoen in [16]. The corresponding modular form is of weight 4 and level 25 and with the trivial character.

(c) Hadamard products

Here we describe how to obtain the unit roots precisely for certain Hadamard products considered in [14, §3]. The only missing piece in loc.cit. is to determine the constant ϵ_4 in Prop 2.7 there. To do this, we study the Frobenius action on the cohomology of the totally degenerate fiber by applying the weight spectral sequence in [12]. For references of Hadamard products and examples of pencils of elliptic curves we discuss here, see [14, §§3.1 and 3.2].

Let $X, Y \to \mathbb{P}^1$ be two pencils of elliptic curves over a finite field k of characteristic p with totally degenerate fibers X_0, Y_0 at 0, respectively. We assume that X_0, Y_0 are strictly normal crossing divisors. Let ξ_1 and ξ_2 be local bases at 0 of horizontal sections of the relative H^1_{cris} of X and Y over $\mathbb{P}^1$, respectively. Then ξ_i are eigenvectors of the relative Frobenius. Denote by c_i the corresponding eigenvalues. Then $c_1 = 1$ if the degenerate curve X_0 is of split multiplicative type over k; $c_1 = -1$ if X_0 is non-split. The corresponding statement for c_2 is similar.

Geometrically, the Hadamard product comes from the following commutative diagram with squares 1, 2, 3 being Cartesian:

$$
\begin{array}{ccccc}
Z & \longrightarrow & \widetilde{X \times Y} & \longrightarrow & X \times Y \\
\downarrow & ① & \downarrow & ② & \downarrow \\
C_1 \cup C_2 & \longrightarrow & \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} & \xrightarrow{b} & \mathbb{P}^1 \times \mathbb{P}^1 \\
\downarrow & ③ & \downarrow & & \downarrow \pi \\
0 & \longrightarrow & \mathbb{P}^1 & = & \mathbb{P}^1.
\end{array}
$$

Here π is the (coordinate-wise) multiplication; b is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ along $(\infty, 0)$ and $(0, \infty)$; C_i are rational curves with $C_1 =$ the strict transformation of $b^{-1}(\mathbb{P}^1 \times 0)$ and $C_2 =$ that of $b^{-1}(0 \times \mathbb{P}^1)$. Notice that the map b induces isomorphisms from C_1 to $\mathbb{P}^1 \times 0$

and from C_2 to $0 \times \mathbb{P}^1$ and $C_1 \cap C_2 = b^{-1}(0,0) \in \widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$. Over $\overline{k}$, write $X_0 = \bigcup D_i$ and $Y_0 = \bigcup E_j$, where D_i and E_j are distinct projective lines. Let $Z^{(i)}$ be the disjoint union of all possible intersections of i distinct irreducible components of Z. We then have

$$\begin{aligned} Z^{(1)} &= \Big(\bigsqcup D_i \times Y\Big) \sqcup \Big(\bigsqcup X \times E_j\Big); \\ Z^{(2)} &= \Big(\bigsqcup (D_i \cap D_r) \times Y\Big) \sqcup \Big(\bigsqcup D_i \times E_j\Big) \sqcup \Big(\bigsqcup X \times (E_j \cap E_s)\Big); \\ Z^{(3)} &= \Big(\bigsqcup (D_i \cap D_r) \times E_j\Big) \sqcup \Big(\bigsqcup D_i \times (E_j \cap E_s)\Big); \\ Z^{(4)} &= \bigsqcup \big((D_i \cap D_r) \times (E_j \cap E_s)\big). \end{aligned} \tag{22}$$

Let us recall the weight spectral sequence ([12, 3.23]; cf. [19, Cor 4.20]):

$$E_1^{-j,i+j} = \bigoplus_{r \geq 0, r \geq -j} H_{cris}^{i-j-2r}(Z^{(1+j+2r)}/W)(-j-r) \Longrightarrow H^i(Z^\times/W^\times),$$

which degenerates at E_2 modulo torsion ([12, Th 3.32]). Here W is the ring of Witt vectors of k and the target $H^i(Z^\times/W^\times)$ is the i-th logarithmic crystalline cohomology of Z. Now assume that X and Y are ordinary and have trivial crystalline cohomology groups of odd degrees. Then the weights of the E_1-terms are all integers and the non-zero terms of E_1 appear only when the weights are even. Thus, putting $E_1^{r,s}$ at the (r,s)-spot, the complete picture of the E_1-terms looks like

wt 6:	$E_1^{-3,6}$	$E_1^{-2,6}$	$E_1^{-1,6}$	$E_1^{0,6}$		
wt 4:		$E_1^{-2,4}$	$E_1^{-1,4}$	$E_1^{0,4}$	$E_1^{1,4}$	
wt 2:			$E_1^{-1,2}$	$E_1^{0,2}$	$E_1^{1,2}$	$E_1^{2,2}$
wt 0:				$E_1^{0,0}$	$E_1^{1,0}$	$E_1^{2,0} \xrightarrow{d} E_1^{3,0}$.

Lemma 4.1 *Let ξ be the Hadamard product of ξ_i and c the eigenvalue of the relative Frobenius action on ξ. Then $c = c_1 c_2$.*

Proof. c represents the relative Frobenius action on the cokernel of d, where d is the boundary map in the displayed E_1-terms above. Let K be the field of fractions of W. By [12, Lemme 5.2], we see that (coker $d) \otimes K$ is 1-dimensional and the Frobenius acts on ξ as the product of its actions on ξ_i. Thus the statement follows. □

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