K3 Surfaces of Finite Height over Finite Fields*†

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Abstract

Arithmetic of K3 surfaces defined over finite fields is investigated. In particular, we show that any K3 surface X of finite height over a finite field k of characteristic $p \geq 5$ has a quasi-canonical lifting Z to characteristic 0, and that for any such Z, the endormorphism algebra of the transcendental cycles $V(Z)$, as a Hodge module, is a CM field over $\mathbb{Q}$. The Tate conjecture for the product of certain two K3 surfaces is also proved. We illustrate by examples how to determine explicitly the formal Brauer group associated to a K3 surface over k. Examples discussed here are all of hypergeometric type.

1 Introduction

We first recall the Tate conjecture for a smooth projective variety X defined over a finite field.

The Tate Conjecture (over finite fields) ([15], Conjecture 1)**.** *Let k be a finite field of characteristic p, and let $\bar{k}$ be an algebraic closure of k. Let $\mathrm{Gal}(\bar{k}/k)$ be the Galois group of $\bar{k}$ over k. Let X be an absolutely irreducible smooth projective variety of dimension d defined over k, and let $X_{\bar{k}} := X \times_k \bar{k}$. Then for any prime $\ell \neq p$ and for any integer $0 \leq r \leq d$, the $\mathrm{Gal}(\bar{k}/k)$-invariant part*

$$H^{2r}_{et}(X_{\bar{k}}, \mathbb{Q}_\ell(r))^{\mathrm{Gal}(\bar{k}/k)}$$

of the étale cohomology group $H^{2r}_{et}(X_{\bar{k}}, \mathbb{Q}_\ell(r))$ is generated by algebraic cycles of codimension r in X defined over k.

Let k be a finite field of characteristic p. Let X be a K3 surface of height h over k. It is known that the Tate conjecture holds true for X over k in the case when $h = 1$ (by Nygaard [10]), or when $h < \infty$ and $p \geq 5$ (by Nygaard and Ogus [11]; see Remark 2.3). The proofs in both cases depend on constructing a good lifting of X to a complex

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projective K3 surface Z, called a "quasi-canonical" lifting, such that the Tate classes in $H^2_{et}(X_{\bar{k}}, \mathbb{Q}_\ell(1))$ are generated by the Hodge classes in $H^2(Z, \mathbb{Q}(1))$ via the comparison

$$H^2(Z, \mathbb{Q}(1)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = H^2_{et}(X_{\bar{k}}, \mathbb{Q}_\ell(1)).$$

Then the validity of the Tate conjecture follows because the Hodge classes in $H^2(Z, \mathbb{Q}(1))$ are generated by divisor classes of Z, and the specialization map to the divisor class group of X (over $\bar{k}$) provides enough algebraic cycles.

In this paper, we will study basic properties of quasi-canonical liftings of K3 surfaces over k of finite height, by making use of the work of Zarhin ([18]) on the Hodge structures for complex K3 surfaces. Let X be a K3 surface over k of finite height h, and let Z be a quasi-canonical lifting of X to characteristic zero. We will prove in Section 2 that the endomorphism algebra of the Hodge structure on the transcendental cycles, $V(Z)$, of Z is a CM field over $\mathbb{Q}$. By studying the Frobenius action on various cohomology groups, we will prove the Tate conjecture for the product of two K3 surfaces in certain cases. This is done in Section 3.

We also study the formal Brauer group $\hat{\mathrm{Br}}(X)$ associated to a K3 surface X (defined in [1]) from the viewpoint of formal group laws. We compute the formal group laws explicitly for several examples including one-parameter families of K3 surfaces. For this, we follow the method of Stienstra ([13]). Examples discussed here are all of hypergeometric type in the sense that the logarithms of (certain liftings of) $\hat{\mathrm{Br}}(X)$ are given by hypergeometric series. As an application, we study congruences and p-adic properties of the coefficients of the logarithms. Our results on formal group laws are stated in Section 4.

Definitions, notations and conventions. If k is a field, $\bar{k}$ denotes an algebraic closure of k. For a prime p, denote $\mathbb{Z}_p$ the p-adic integers and $\mathbb{Q}_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Q}$. Throughout the paper, $\mathbb{F}_q$ denotes a fixed finite field of q elements. We also fix an algebraic closure of $\mathbb{F}_q$ implicitly.

Suppose σ is a linear endomorphism on a finite dimensional vector space V over a field k. The *reciprocal characteristic polynomial of σ on V* is the polynomial

$$\det(1 - T\sigma | V)$$

with indeterminate T. It has coefficients in k.

Let k be a field. A smooth, geometrically irreducible, projective variety X of dimension 2 over k is called a *K3 surface* if the canonical line bundle of X is isomorphic to the structure sheaf $\mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

Let X be a K3 surface defined over a field k. We fix the following notations.

• $\mathrm{NS}(X)$ = the Néron-Severi group of X over k = the classes of divisors over k on X modulo rational equivalence.

• $\rho(X)$ = the rank of $\mathrm{NS}(X)$ as an abelian group.

• Let k' be a field extension of k and R be a local ring with residue field k'. A *lifting Y over R of X* is a flat scheme Y over R such that $Y \times_R k' = X \times_k k'$.

• Suppose k is a perfect field of characteristic $p > 0$. Let $\hat{\mathrm{Br}}(X)$ denote the formal Brauer group of X. It is known (see [1], Examples in p.90) that $\hat{\mathrm{Br}}(X)$ is representable by a one-dimensional commutative formal group. Recall that the *height* of X is defined to be the height of $\hat{\mathrm{Br}}(X)$ as a formal group. Let h denote the height of X. Then h can take any integer between 1 and 10, or ∞ (loc.cit.). In the former case, X is said to have *finite height*. In particular, if $h = 1$, X is said to be *ordinary*. X is called *supersingular* if $h = \infty$ (i.e., $\hat{\mathrm{Br}}(X)$ is the additive formal group).

2 A quasi-canonical lifting to characteristic zero

We fix a finite field $k = \mathbb{F}_q$ of characteristic p, and also fix a K3 surface X defined over k. The following definition is a reformulated version (for a K3 surface) of the definition of quasi-canonical liftings given in [11], Definition (1.5). The formulation given here is customized in more suitable ways for our discussion on K3 surfaces.

Definition 2.1. A projective K3 surface Z over $\mathbb{C}$ is called a *quasi-canonical lifting* of X if there exist

(a) a complete discrete valuation ring $R \subset \mathbb{C}$ with residue field $k = \mathbb{F}_q$,

(b) a K3 surface Y defined over R with $Y_{\mathbb{C}} = Z$, which lifts X, and

(c) an endomorphism σ on $H^2(Z, \mathbb{Q})$ respecting the Hodge structure (i.e., a Hodge cycle in $\mathrm{End}_{\mathbb{Q}} H^2(Z, \mathbb{Q})$),

satisfying the following two conditions:

(1) Under the natural identifications

$$H^2(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = H^2(Y_{\mathbb{C}}, \mathbb{Q}_\ell) = H^2_{et}(X_{\bar{k}}, \mathbb{Q}_\ell),$$

the endomorphism σ coincides with the geometric Frobenius on $H^2_{et}(X_{\bar{k}}, \mathbb{Q}_\ell)$ for any prime $\ell \neq p$. Similarly, under the natural identifications ([2], 6.1.4)

$$H^2(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = H^2_{dR}(Y/R) \otimes_R \mathbb{C} = H^2_{cris}(X/W) \otimes_W \mathbb{C},$$

the endormorphism σ coincides with the Frobenius endomorphism on the crystalline cohomology $H^2_{cris}(X/W)$. Here W is the ring of Witt vectors of k with the embedding $W \to R \to \mathbb{C}$.

(2) There exists an integer n such that the fixed part of $(\sigma/q)^n$ on $H^2(Z, \mathbb{Q})$ is precisely equal to the rational Néron–Severi group $\mathrm{NS}(Z)_{\mathbb{Q}} := \mathrm{NS}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$, and the specialization map

$$\mathrm{NS}(Z)_{\mathbb{Q}} = \mathrm{NS}(Y_{\mathbb{C}})_{\mathbb{Q}} \to \mathrm{NS}(X_{\bar{k}})_{\mathbb{Q}}$$

is an isomorphism.

We simply call the Hodge cycle σ in (c) the *lifted geometric Frobenius* on Z.

Notice that when condition (1) above is fulfilled, condition (2) implies the validity of the Tate conjecture for X over k.

The Tate conjecture for a K3 surface of finite height was established by Nygaard and Ogus [11].

Theorem 2.2 (Nygaard and Ogus [11]). *Let $k = \mathbb{F}_q$ be a finite field of characteristic $p \geq 5$. Let X be a K3 surface of finite height over k. Then there exist a totally ramified finite extension R of the Witt ring $W(k)$ of k and a lifting Y over R of X such that for any embedding $R \to \mathbb{C}$, the induced complex K3 surface $Z = Y_{\mathbb{C}}$ is a quasi-canonical lifting of X. In particular, the Tate conjecture is true for X over k.*

Proof. Write W for $W(k)$. By [11], Theorem (5.6), there exist a totally ramified finite extension R of W and a lifting Y over R of X such that the geometric Frobenius σ on $H^2_{cris}(X/W)$ preserves the Hodge filtration induced from Y through the identification

$$H^2_{cris}(X/W) \otimes_W L = H^2_{dR}(Y/R) \otimes_R L.$$

Here L is the field of fractions of R. Then by op.cit., Proposition (1.7), for any embedding $R \to \mathbb{C}$, the map σ preserves the rational structure $H^2(Z, \mathbb{Q})$ via the identification

$$H^2(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = H^2_{dR}(Y/R) \otimes_R \mathbb{C}.$$

Moreover, this induced map σ preserves the Hodge structure on $H^2(Z, \mathbb{Q})$. By the construction, σ on $H^2(Z, \mathbb{Q})$ satisfies condition (1) in Definition 2.1. Furthermore, σ also satisfies condition (2) in Definition 2.1 (op.cit., Theorem (2.1) and Remark (2.2.3)). This completes the proof. □

Remark 2.3. The construction of a quasi-canonical lifting of X in [11] rests on a fact that there is an equivalence between the category of the deformations of a K3 surface of finite height, and that of the deformations of the associated crystal (see [11], Theorem (4.5)). The assumption that $p \geq 5$ is needed to establish that the functor between these two categories is fully faithful (op.cit., Lemma (4.6)). At this moment, we do not know if there always exists a quasi-canonical lifting of X when $p = 2$ or 3.

(2.4) Here we recall some results from Zarhin [18]. For any complex projective K3 surface Z, we define the *transcendental cycles* of Z to be the orthogonal complement of $\mathrm{NS}(Z)$ in $H^2(Z, \mathbb{Q}(1))$ with respect to the cup-product:

$$V(Z) := \mathrm{NS}(Z)^{\perp}_{\mathbb{Q}} \subset H^2(Z, \mathbb{Q}(1)).$$

Since $\mathrm{NS}(Z)$ is contained in the (0,0)-part in the Hodge decomposition of $H^2(Z, \mathbb{Q}(1))$, the transcendental part $V(Z)$ inherits a rational Hodge structure from $H^2(Z, \mathbb{Q}(1))$. Moreover, $V(Z)$ is an irreducible Hodge structure ([18], Theorem 1.4.1). Let

$$\mathcal{E} = \mathrm{End}_{\mathrm{Hdg}}\, V(Z)$$

be the set of linear endomorphisms of $V(Z)$ that respect the Hodge structure. Then $\mathcal{E}$ is either a totally real field or a CM field (op.cit., Theorems 1.5.1 and 1.6). With respect to the structure morphism $Z \to \mathrm{Spec}\, \mathbb{C}$, there is a natural embedding of $\mathcal{E}$ into $\mathbb{C}$ given by

$$\mathcal{E} \to \mathrm{End}(V(Z) \otimes \mathbb{C}) \to \mathrm{End}\, H^0(Z, \Omega^2) = \mathbb{C}, \tag{1}$$

where the second map is the projection via the Hodge decomposition.

Corollary 2.5. *Let $k = \mathbb{F}_q$ be a finite field of characteristic $p \geq 5$. Let X be a K3 surface of finite height over k. Then for any quasi-canonical lifting Z over $\mathbb{C}$ of X, the endomorphism algebra $\mathcal{E}$ of the transcendental cycles $V(Z)$, as a Hodge module, is a CM field over $\mathbb{Q}$.*

Proof. By Theorem 2.2, the Tate conjecture is true for X over k. Since $\sigma \in \operatorname{End} V(Z)$ is a Hodge cycle, $\mathbb{Q}[\sigma] \subset \mathcal{E}$. Notice that the element σ can not be totally real, for otherwise a characteristic root of σ on $V(Z)$ would be a root of unity. This contradicts the condition (2) in Definition 2.1. Thus $\mathcal{E}$ must be a CM field. □

3 The Frobeius on the transcendental part

Definition 3.1 (cf. [19], 2.0.1)**.** Let X be a K3 surface over $k = \mathbb{F}_q$. Let $A_\ell(X)$ be the set of elements $\alpha \in H^2_{et}(X_{\bar{k}}, \mathbb{Q}_\ell(1))$ such that α is invariant under $\mathrm{Gal}(\bar{k}/k')$ for some finite extension k' of k (depending on α). We call $A_\ell(X)$ the *algebraic part* of the étale cohomology $H^2_{et}(X_{\bar{k}}, \mathbb{Q}_\ell(1))$ of $X_{\bar{k}}$. Let $V_\ell(X)$ be the orthogonal complement of $A_\ell(X)$ with respect to the cup-product. We call $V_\ell(X)$ the *transcendental part* of $H^2_{et}(X_{\bar{k}}, \mathbb{Q}_\ell(1))$. (Cf. [19], Remark 3.3.3.) We have the decomposition

$$H^2_{et}(X_{\bar{k}}, \mathbb{Q}_\ell(1)) = V_\ell(X) \oplus A_\ell(X).$$

We will write $R(X,T)$ for the reciprocal characteristic polynomial of the geometric Frobenius σ on $V_\ell(X)$. That is,

$$R(X,T) = \det\left(1 - T\sigma | V_\ell(X)\right).$$

If we let $F(X,T)$ be the reciprocal characteristic polynomial of σ on $H^2_{et}(X_{\bar{k}}, \mathbb{Q}_\ell(1))$, and put

$$R'(X,T) = \frac{F(X,T)}{R(X,T)}.$$

Then the zeros of $R'(X,T)$ consists of zeros of $F(X,T)$ which are roots of unity. Moreover, we have

$$R(X,qT) \in 1 + T\mathbb{Z}[T].$$

This is because all reciprocal roots of $R(X,qT)$ are algebraic integers (cf. [19], Remark 2.0.3, 2.1). If the Tate conjecture is true for X over k, then $R(X,T)$ is of degree $22 - \rho(X_{\bar{k}})$, which is the case when k is of characteristic $p \geq 5$ and X is of finite height (see (2.2)).

Proposition 3.2. *Let $k = \mathbb{F}_q$ be a finite field of characteristic p. Let X be a K3 surface of finite height h over k. Then $R(X,T)$ is a power of a $\mathbb{Q}$-irreducible polynomial of the form:*

$$R(X,T) = Q(X,T)^r$$

where $Q(X,T)$ is a $\mathbb{Q}$-irreducible polynomial with constant term 1, and the exponent r divides h.

Proof. We give two proofs here. The first method uses the formal Brauer group $\hat{\mathrm{Br}}(X)$ associated to X, and the second method is by lifting X to characteristic zero when $p \geq 5$.

(a) Suppose that

$$R(X,T) = Q_1(T) \cdots Q_r(T)$$

is the decomposition of $R(X,T)$ into irreducible polynomials with contant terms 1 over $\mathbb{Q}$. Then over $\mathbb{Q}_p$, each $Q_i(T)$ decomposes as a product

$$Q_i^{<0}(T) Q_i^0(T) Q_i^{>0}(T)$$

according to the slopes of the Newton polygon of $Q_i(T)$, e.g., $Q_i^{<0}(T)$ denotes the polynomial corresponding to the Newton slope < 0 part, and respectively for $Q_i^0(T)$ and $Q_i^{>0}(T)$. By the symmetry of the Newton polygon,

$$Q_i^{>0}(T) = (-1)^{d_i} c_i T^{d_i} Q_i^{<0}(1/T),$$

where d_i is the degree of $Q_i^{<0}(T)$ and c_i is the product of roots of $Q_i^{<0}(T)$. Therefore, for each i, the factor $Q_i^{<0}(T) \neq 1$, for otherwise $Q_i(T) = Q_i^0(T)$ and roots of $Q_i(T) = Q_i^0(T)$ are roots of unity.

On the other hand, the product

$$Q_1^{<0}(qT) \cdots Q_r^{<0}(qT)$$

is the reciprocal characteristic polynomial of the Frobenius endomorphism on the Cartier module of $\hat{\mathrm{Br}}(X)$, which is equal to $(Q^{<0}(qT))^r$ for some irreducible polynomial $Q^{<0}(qT)$ over $\mathbb{Z}_p$ (see [4], Theorem (24.2.6)). Thus for all i, $Q_i(T)$ must be equal to each other.

(b) Assume that k is of characteristic $p \geq 5$. Let Z be a quasi-canonical lifting of X to $\mathbb{C}$ and $V(Z) \subset H^2(Z, \mathbb{Q}(1))$ be the transcendental cycles of Z. Then the Frobenius endomorphism σ can be regarded as a Hodge cycle in the endomorphism algebra $\mathcal{E}$ of the Hodge structure $V(Z)$. Since $\mathcal{E}$ is a field, the minimal polynomial $m(T)$ of σ on $V(Z)$ has only simple roots. Thus $m(T)$ is irreducible over $\mathbb{Q}$. Moreover, since $V(Z)$ is also a vector space over $\mathbb{Q}[\sigma] \simeq \mathbb{Q}[T]/m(T)$, the characteristic polynomial of σ on the $\mathbb{Q}$-vector space $V(Z)$ is then the r-th power of $m(T)$ where $r = \dim_{\mathbb{Q}[\sigma]} V(Z)$. Therefore the assertion follows. □

(3.3) Suppose that X is of finite height h. Let $\tau = \dim_{\mathbb{Q}_\ell} V_\ell(X)$. Notice that the splitting field of the polynomial $Q(X,T)$ of degree (τ/r) is a CM-field (see the proof of Corollary 2.5). Thus τ/r is an even integer. All possible values for r in various situations are tabulated below. Notice that when X is ordinary (i.e., $h = 1$), the polynomial $R(X,T)$ is always irreducible over $\mathbb{Q}$. This is a result of Zarhin ([19], Theorem 1.1) and is used in his proof of the Tate conjecture for powers of an ordinary K3 surface over a finite field.

$\tau\backslash h$	1	2	3	4	5	6	7	8	9	10
2	1									
4	1	1, 2								
6	1	1	1, 3							
8	1	1, 2	1	1, 2, 4						
10	1	1	1	1	1, 5					
12	1	1, 2	1, 3	1, 2	1	1, 2, 3, 6				
14	1	1	1	1	1	1	1, 7			
16	1	1, 2	1	1, 2, 4	1	1, 2	1	1, 2, 4, 8		
18	1	1	1, 3	1	1	1, 3	1	1	1, 3, 9	
20	1	1, 2	1	1, 2	1, 5	1, 2	1	1, 2	1	1, 2, 5, 10

As an application of Proposition 3.2, we have the following consequence, which is communicated to us by Y.G. Zarhin.

Corollary 3.4. *Let X_1 and X_2 be two K3 surfaces of finite height over $k = \mathbb{F}_q$. Suppose that the heights of X_i are different. Then the Tate conjecture is true for the product $X_1 \times X_2$ over k.*

Proof. It suffices to show that the tensor product $V_\ell(X_1) \otimes V_\ell(X_2)$ of the transcendental parts $V_\ell(X_i)$ of X_i does not create non-trivial Tate classes. For $i = 1, 2$, let $R(X_i, T) = Q(X_i, T)^{r_i}$, where $Q(X_i, T)$ are $\mathbb{Q}$-irreducible polynomials with constant terms 1. Notice that $Q(X_1, T)$ and $Q(X_2, T)$ have different Newton polygons by the assumption. In particular $Q(X_1, T) \neq Q(X_2, T)$.

Suppose that there is a non-trivial Tate class in $V_\ell(X_1) \otimes V_\ell(X_2)$. Then by replacing k by a finite extension, we may assume that there exists a root α_i of $Q(X_i, T)$ $(i = 1, 2)$ such that $\alpha_1\alpha_2 = 1$. Thus $\alpha_2 = \alpha_1^{-1}$. Since all the roots of $Q(X_2, T) \in \mathbb{Q}[T]$ have complex norm one, $Q(X_2, \alpha_1^{-1}) = 0$ implies that $Q(X_2, \alpha_1) = 0$. So $Q(X_1, T)$ and $Q(X_2, T)$ have a common root. Now the irreducibility of both polynomials $Q(X_1, T)$ and $Q(X_2, T)$ with the same constant term 1 implies that $Q(X_1, T) = Q(X_2, T)$, which is a contradiction. □

Lemma 3.5. *Let $k = \mathbb{F}_q$ be a finite field of characteristic $p \geq 5$. Let X be a K3 surface of finite height over k. Suppose that $R(X, T)$ is irreducible over $\mathbb{Q}$. Then the Tate conjecture is true for the self-product $X \times X$ of X over k.*

Proof. This is a special case of [20], Theorem 4.4. (See also op.cit., Lemma 2.1.) Since $R(X, T)$ is irreducible, it implies that the vector space of Galois invariants

$$(V_\ell(X) \otimes V_\ell(X))^{\mathrm{Gal}(\bar{k}/k)} \subset V_\ell(X) \otimes V_\ell(X)$$

is generated by the linearly independent set $\{id, \sigma, \ldots, \sigma^{\tau-1}\}$, where $\tau = 22 - \rho(X_{\bar{k}})$. □

Corollary 3.6. *Suppose that X is a K3 surface of finite height defined over the prime field $k = \mathbb{F}_p$ with $p \geq 5$. Then the Tate conjecture is true for $X \times X$ over k.*

Proof. It suffices to show that $R(X,T)$ is irreducible over $\mathbb{Q}$. Notice that $R(X,T)$ is irreducible if the action of the Frobenius σ on the Cartier module of $\hat{\mathrm{Br}}(X)$ is irreducible (see the proof (a) of Proposition 3.2). Under the assumption, $\mathbb{Q}_p[\sigma]$, regarded as in $(\mathrm{End}_{\bar{k}}\hat{\mathrm{Br}}(X))\otimes\mathbb{Q}_p$, is a totally ramified field extension of degree h over $\mathbb{Q}_p$, where h is the height of $\hat{\mathrm{Br}}(X)$ ([4], Remark (24.2.7)). Thus $R(X,T)$ is irreducible over $\mathbb{Q}$. The assertion now follows from the lemma above. □

Remark 3.7. Let X be a K3 surface of finite height over $k=\mathbb{F}_q$. As in Lemma 3.5, we assume that $R(X,T)$ is irreducible. If the Zariski closure of the cyclic group generated by σ/q is equal to the group, U, of all elements of norm 1 in $\mathbb{Q}[\sigma]$, then we can use the same argument as in [20], Theorem 6.1 (ii) to show that the Tate conjecture holds true for the n-th power $X^n := X\times X\times\cdots\times X$ of X over k for any positive integer n. However, we are not able to determine if σ/q generates U or not. We thank Zarhin for pointing out to us that the irreducibility of $R(X,T)$ is not enough to guarantee the Tate conjecture for the n-th power X^n of X when $n>2$.

4 Formal group laws

As the formal Brauer group $\hat{\mathrm{Br}}(X)$ arising from a K3 surface X over a finite field is one-dimensional, it must be associated to a certain Dirichlet series ([5]). We will discuss $\hat{\mathrm{Br}}(X)$ from the point of view of formal group *laws*, along the line of Honda [5].

(4.1) Let $k=\mathbb{F}_q$ be a finite field of characteristic p. Let $W(k)$ be the ring of Witt vectors of k. Let X be a K3 surface over k. Let $P(X,T)$ be the reciprocal characteristic polynomial of the Frobenius endomorphism on the crystalline cohomology $H^2_{cris}(X/W)$. Then $P(X,T)\in 1+T\cdot\mathbb{Z}[T]$, and over $\mathbb{Z}_p$, the polynomial $P(X,T)$ has a natural slope decomposition

$$P(X,T)=P_{<1}(X,T)\cdot P_1(X,T)\cdot P_{>1}(X,T),$$

where $P_{<1}$ denotes the polynomial corresponding to the Newton slope less than 1 part, and respectively for P_1 and $P_{>1}$. If X is of finite height h, then $P_{<1}$ is of degree h and

$$P_{>1}(X,T)=cT^h\cdot P_{<1}(X,q^2/T)$$

for some constant $c\in\mathbb{Z}_p$. If X is supersingular, $P_{<1}=P_{>1}=1$.

Theorem 4.2. *Let X be a K3 surface over $k=\mathbb{F}_q$. Let*

$$P(X,T)=P_{<1}(X,T)\cdot P_1(X,T)\cdot P_{>1}(X,T)$$

be the slope decomposition over $\mathbb{Z}_p$ of the reciprocal characteristic polynomial of the geometric Frobenius on $H^2_{cris}(X/W)$. Then the isomorphism class of $\hat{\mathrm{Br}}(X)$ over k is determined by $P_{<1}(X,T)$.

Proof. Assume that X is of height $h<\infty$. Let $R_{<1}(T)$ be the minimal reciprocal polynomial of the Frobenius endomorphism of $\hat{\mathrm{Br}}(X)$ on its Cartier module. Then $R_{<1}(T)$ is irreducible over $\mathbb{Z}_p$ (see [4], Theorem (24.2.6)) and

$$P_{<1}(X,T)=R_{<1}(T)^r$$

for some positive integer r. One knows that the formal group $\hat{\mathrm{Br}}(X)$ is determined by $R_{<1}(T)$ (op.cit., Proposition (24.2.9)), and thus it is determined by $P_{<1}(X,T)$.

On the other hand, if X is supersingular, then $P_{<1}=1$ and $\hat{\mathrm{Br}}(X)$ is isomorphic to the additive formal group $\hat{\mathbb{G}}_a$. The assertion is trivial in this case. $\square$

(4.3) In what follows, we let $k=\mathbb{F}_q$ be the finite field of q elements of characteristic p, and let $W=W(k)$ be the ring of Witt vectors of k.

Here we compute explicitly the formal group laws that realize the formal Brauer groups $\hat{\mathrm{Br}}(X)$ of certain K3 surfaces X over k. The explicit realizations are obtained based on the work of Stienstra [13].

The examples below involve hypergeometric series. This seems to come from the fact that the principle part of the defining equation for a K3 surface discussed in our examples is very symmetric with respect to the variables, and the parameter t appears in the product of the variables. We recall that the hypergeometric series ${}_mF_n$ with upper parameters $\{a_i\}_{i=1}^m$ and lower parameters $\{b_i\}_{i=1}^n$, $a_i, b_i\in\mathbb{C}$ and $b_i\notin\mathbb{Z}_{\leq 0}$, is defined by the formal power series

$$ {}_mF_n\left(\begin{array}{c} a_1, a_2, \cdots, a_m \\ b_1, b_2, \cdots, b_n \end{array} ; x\right)=\sum_{r=0}^{\infty}\frac{(a_1)_r(a_2)_r\cdots(a_m)_r}{(b_1)_r(b_2)_r\cdots(b_n)_r}\cdot\frac{x^r}{r!}, $$

where $(a)_0=1$ and $(a)_r=a(a+1)\cdots(a+r-1)$ for $r>0$ is the *Pochhammer symbol.* Notice that if a_i is a non-positive integer for some i, then the hypergeometric series ${}_mF_n$ is a polynomial. We may regard that the series is defined over a ring R whenever the expansion makes sense over R.

(4.4) Elliptic modular K3 surfaces.

(4.4.1) (Shioda [12]) Let

$$ \Gamma(4)=\left\{\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \mid c\equiv 0 \pmod 4\right\} $$

be the principal congruence subgroup of $\Gamma=\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ of level 4. Then $\Gamma(4)$ gives rise to an elliptic modular K3 surface X over a finite field $k=\mathbb{F}_q$ of characteristic $p>2$ with $q=p^a$. More precisely, X is the minimal model over k of the Jacobi quartic

$$ y^2=(1-\sigma^2x^2)(1-\sigma^{-2}x^2), $$

which is defined over the function field $K=k(\sigma)$ with σ a variable over k. Suppose $\sqrt{-1}\in k$. Then the zeta-function of X over k is given as follows:

$$ Z(X,T)=\frac{1}{(1-T)(1-qT)^{20}(1-q^2T)}\cdot\frac{1}{H_{3,q}(T)}. $$

Here $H_{3,q}(T)$ is the Hecke polynomial

$$ H_{3,q}(T)=\begin{cases} 1-(\pi^2+\pi'^2)T+q^2T^2 & \text{if} \quad p\equiv 1 \pmod 4 \\ (1-qT)^2 & \text{if} \quad p\equiv 3 \pmod 4 \end{cases} $$

where π, π' are integers in $\mathbb{Z}[\sqrt{-1}]$ such that $\pi = \alpha^a$ and $\pi = \alpha'^a$ for some $\alpha, \alpha' \in \mathbb{Z}[\sqrt{-1}]$ with $\alpha\alpha' = p$ and $\alpha \equiv 1 \pmod{2\sqrt{-1}}$ ([12], Example A.10). Fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. When $p \equiv 1 \pmod 4$, assume that π is a p-adic unit with respect to this embedding.

(4.4.2) Make the change of variable $x \mapsto \sigma x$ in the defining equation for the Jacobi quartic. Then this leads to the elliptic pencil given by

$$y^2 = (1 - \sigma^4 x^2)(1 - x^2)$$

with parameter σ. Consider the formal group law $\hat{G}$ over $\mathbb{Z}$ with the logarithm

$$l(\tau) = \sum_{m=0}^{\infty} a(m) \frac{\tau^{4m+1}}{4m+1},$$

where

$$\begin{aligned} a(m) &= \binom{2m}{m} \cdot {}_2F_1\left(\begin{matrix} -m, -m \\ 1 \end{matrix} ; 1 \right) \\ &= \binom{2m}{m}^2. \end{aligned}$$

The last equality follows from the Vandermonde involution formula. Then the base change to k of $\hat{G}$ realizes $\hat{\mathrm{Br}}(X)$ ([13], Theorem 2).

Suppose $p = 4m + 1$. Then X is of height one. Take $k = \mathbb{F}_p$ to be the prime field. Then for any integer $s \geq 0$, we have ([14], Theorem (A.8)(v))

$$\frac{a\left(\frac{p^{s+1}-1}{4}\right)}{a\left(\frac{p^s-1}{4}\right)} \equiv \pi^2 \pmod{p^{s+1}},$$

where π^2 is the p-adic unit reciprocal root of $H_{3,q}(T), q = p$ discussed in (4.4.1). On the other hand, since the polynomials

$${}_2F_1\left(\begin{matrix} -m_s, -m_s \\ 1 \end{matrix} ; 1 \right),$$

with $m_s := \frac{1}{4}(p^s - 1)$, converge p-adically to

$${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ 1 \end{matrix} ; 1 \right),$$

one knows that

$$\lim_{s \to \infty} \frac{a\left(\frac{p^{s+1}-1}{4}\right)}{a\left(\frac{p^s-1}{4}\right)} = h(1)^2.$$

Here $h(x)$ is the formal power series

$$h(x) = {}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ 1 \end{matrix} ; x \right) \Big/ {}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ 1 \end{matrix} ; x^p \right)$$

with coefficients in $\mathbb{Z}_p$, and it converges p-adically at $x = 1$. Furthermore, since

$$-\frac{1}{4} = \sum_{i=0}^{\infty} mp^i,$$

we have

$$h(1) = \frac{\Gamma_p(\frac{1}{4})^2}{\Gamma_p(\frac{1}{2})},$$

where $\Gamma_p(x)$ is the Morita's p-adic gamma-function ([7], Theorem 2). Thus

$$\pi^2 = \frac{\Gamma_p(\frac{1}{4})^4}{\Gamma_p(\frac{1}{2})^2}.$$

This is a consequence of the Gross-Koblitz formula (see [16], (F.2) combined with some basic equalities for $\Gamma_p(x)$ in [8], p.42).

If $p \equiv -1 \pmod 4$, then X is supersingular. Thus $l(\tau) \in W[[\tau]]$, that is,

$$p^{2s} \;\Big|\; a\left(\frac{p^{2s}-1}{4}\right),$$

which implies that

$$p^s \;\Big|\; \binom{\frac{p^{2s}-1}{2}}{\frac{p^{2s}-1}{4}}$$

for all non-negative integers s.

(4.5) Examples: Diagonal quartic K3 surfaces

(4.5.1) Consider the twisted diagonal quartic surface X over k defined by

$$c_1T_1^4 + c_2T_2^4 + c_3T_3^4 + c_4T_4^4 = 0 \subset \mathbb{P}_k^3$$

with $c_i \in k$ and $c = c_1c_2c_3c_4 \neq 0$. Let $\hat{c}_i \in W$ be a lifting of c_i for each i and let

$$\hat{c} = \hat{c}_1\hat{c}_2\hat{c}_3\hat{c}_4.$$

Put

$$l(\tau) = \sum_{n=0}^{\infty} \hat{c}^n \frac{(4n)!}{(n!)^4} \frac{\tau^{4n+1}}{4n+1}$$

and define

$$\hat{G}(\tau_1, \tau_2) = l^{-1}(l(\tau_1) + l(\tau_2)).$$

Then $\hat{G}$ is a formal group (defined over W) whose reduction G to k realizes $\widehat{\mathrm{Br}}(X)$ ([13], Theorem 1).

Now let $k = \mathbb{F}_q$ be the finite field with $q = p^a$ elements of characteristic p. The formal group $G(\tau_1, \tau_2)$ is of multiplicative type (i.e., G is of height 1, and hence G is isomorphic over $\bar{k}$ to $\hat{\mathbb{G}}_m$) if $p \equiv 1 \pmod 4$, and is additive (i.e., G has infinite height and hence G is isomorphic to $\hat{\mathbb{G}}_a$) if $p \equiv 3 \pmod 4$. In the later case, it means that $l(\tau)$ is an element in $W[[\tau]]$. Thus it implies that for all $n > 1$, we have

$$\frac{(4n)!}{(n!)^4} \equiv 0 \pmod{p^s},$$

where s is the largest integer such that $p^s \mid (4n+1)$.

On the other hand, assume that $p \equiv 1 \pmod 4$. Write

$$a(n) = \hat{c}^n \frac{(4n)!}{(n!)^4}.$$

Then there exists an element $\alpha \in W$ such that

$$a\left(\frac{\mu p^{s+1}-1}{4}\right) \equiv \alpha \cdot a\left(\frac{\mu p^s - 1}{4}\right) \pmod{p^{s+1}} \tag{2}$$

for all non-negative integers μ, s (with the convention that $a(n) = 0$ if n is not an integer) ([14], Theorem (A.8)(v)). Let

$$\pi = \alpha^{1+\sigma+\cdots+\sigma^{a-1}}.$$

Here the upper script σ indicates the action by the absolute Frobenius σ on W. Then π turns out to be a twisted Jacobi sum ([3], Lemma 3.4) and equation (2) gives p-adic approximations of π.

(4.5.2) Let p be an odd prime. Generalizing the above example, we consider the one-parameter family of K3 surfaces X_λ given by one of the following quadrics :

$$c_1T_1^4 + c_2T_2^4 + c_3T_3^4 + c_4T_4^4 - 4\lambda T_1T_2T_3T_4 = 0,$$

$$c_1T_1^3T_2 + c_2T_2^3T_3 + c_3T_3^3T_4 + c_4T_4^3T_1 - 4\lambda T_1T_2T_3T_4 = 0,$$

$$c_1T_1^2T_2T_3 + c_2T_2^2T_3T_4 + c_3T_3^2T_4T_1 + c_4T_4^2T_1T_2 - 4\lambda T_1T_2T_3T_4 = 0,$$

with $c_i \in k$, $c = c_1c_2c_3c_4 \neq 0$ and parameter λ. We only consider those values of λ such that the corresponding surfaces X_λ are smooth. Let $\hat{c_i} \in W$ be a lifting of c_i for each i, and let

$$\hat{c} = \hat{c_1}\hat{c_2}\hat{c_3}\hat{c_4} \in W.$$

Also take a lifting $\hat{\lambda} \in W$ of λ. Put

$$l(\tau) = \sum_{m=0}^{\infty} a(m) \frac{\tau^{m+1}}{m+1},$$

where

$$a(m) = (-4\hat{\lambda})^m {}_4F_3\left(\begin{matrix} \frac{-m}{4}, \frac{-m+1}{4}, \frac{-m+2}{4}, \frac{-m+3}{4} \\ 1, 1, 1 \end{matrix} ; \hat{c}\hat{\lambda}^{-4}\right).$$

Define

$$G_{\hat{\lambda}}(\tau_1, \tau_2) = l^{-1}(l(\tau_1) + l(\tau_2)).$$

Then $G_{\hat{\lambda}}$ is a formal group (defined over W) whose reduction G_λ to k realizes $\hat{\mathrm{Br}}(X_\lambda)$ ([13], Theorem 1).

If $a(p-1) \equiv 0 \pmod{p}$, then G_λ is isomorphic to the additive group $\hat{\mathbb{G}}_a$. Thus in this case, for any $m > 0$,

$$a(m) \equiv 0 \pmod{p^s}$$

where s is the largest integer such that $p^s \mid (m+1)$.

On the other hand, for those λ satisfying $a(p-1) \not\equiv 0 \pmod{p}$, the formal group G_λ is of multiplicative type, i.e., is isomorphic over $\bar{k}$ to the multiplicative group $\hat{\mathbb{G}}_m$. In this case, we take $\hat{c}$ and $\hat{\lambda} \in W$ to be the Teichmüller lifting of c and λ, respectively. Then for each such λ, there exists a p-adic unit $\alpha_\lambda \in W$ such that

$$a(\mu p^{s+1} - 1) \equiv \alpha_\lambda \cdot a(\mu p^s - 1)^\sigma \pmod{p^{s+1}}$$

for any integers $\mu, s \geq 0$ ([14], Theorem (A.8)(v)). Furthermore, if $q = p^a$, the element

$$\pi_\lambda = \alpha_\lambda^{1+\sigma+\cdots+\sigma^{a-1}}$$

is the unique p-adic unit root of the characteristic polynomial of the geometric Frobenius on $H^2_{et}((X_\lambda)_{\bar{k}}, \mathbb{Q}_\ell)$. It is then easy to see that the polynomials

$${}_4F_3\left(\begin{matrix} \frac{-m_s}{4}, \frac{-m_s+1}{4}, \frac{-m_s+2}{4}, \frac{-m_s+3}{4} \\ 1, 1, 1 \end{matrix} ; x \right)$$

with $m_s := (p^s - 1)$, converge p-adically to the hypergeometric series

$${}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \\ 1, 1 \end{matrix} ; x \right) = \sum_{r=0}^{\infty} 4^{-4r} \binom{4r}{r} \binom{3r}{r} \binom{2r}{r} x^r.$$

(Notice that this formal power series ${}_3F_2$ has coefficients in $\mathbb{Z}[\frac{1}{2}]$.) Therefore

$$\begin{aligned} \alpha_\lambda &= \lim_{s\to\infty} \frac{a(p^{s+1}-1)}{a(p^s-1)^\sigma} \\ &= \hat{\lambda}^{p-1} f(\hat{c}\hat{\lambda}^{-4}). \end{aligned}$$

Here $f(x)$ is the formal power series

$$f(x) = {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \\ 1, 1 \end{matrix} ; x \right) \Big/ {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \\ 1, 1 \end{matrix} ; x^p \right) \tag{3}$$

with coefficients in $\mathbb{Z}_p$, and it converges p-adically at $x = \hat{c}\hat{\lambda}^{-4}$. Note that the requirement of $\hat{c}\hat{\lambda}^{-4}$ being the Teichmüller lifting of $c\lambda^{-4}$ is needed in order to have the compact formula (3) for $f(x)$.

(4.6) Examples: Double sextic K3 surfaces

(4.6.1) Let X be the double cover of a smooth sextic defined in $\mathbb{P}^1 \times \mathbb{P}^2$ over k by

$$Y^2 = c_1 T_1^6 + c_2 T_2^6 + c_3 T_3^6$$

with $c_i \in k$ and $c = c_1 c_2 c_3 \neq 0$. Let $\hat{c_i} \in W$ be a lifting of c_i for each i and let

$$\hat{c} = \hat{c_1}\hat{c_2}\hat{c_3},$$

which is a lifting of c to W. Put

$$l(\tau) = \sum_{n=0}^{\infty} \hat{c}^n \frac{(3n)!}{(n!)^3} \frac{\tau^{6n+1}}{6n+1}$$

and

$$\hat{G}(\tau_1, \tau_2) = l^{-1}(l(\tau_1) + l(\tau_2)).$$

Then $\hat{G}$ is a formal group (defined over W) whose reduction G to k realizes $\hat{\mathrm{Br}}(X)$ ([13], Theorem 2).

Over the finite field k of characteristic p, the formal group $G(\tau_1, \tau_2)$ is isomorphic over $\bar{k}$ to the multiplicative group $\hat{\mathbb{G}}_m$ if $p \equiv 1 \pmod 6$, and to the additive group $\hat{\mathbb{G}}_a$ otherwise. In the later case, it follows that $l(\tau)$ is an element in $W[[\tau]]$. Thus it implies that for all $n > 1$, we have

$$\frac{(3n)!}{(n!)^3} \equiv 0 \pmod{p^s}$$

where s is the largest integer such that $p^s \mid (6n+1)$.

(4.6.2) More generally, let $X_\lambda, \lambda \neq 0$ be the one-parameter family of K3 surfaces given by the double cover of $\mathbb{P}^2$ ramified along any one of the following curves :

$$c_1 T_1^6 + c_2 T_2^6 + c_3 T_3^6 - 3\lambda T_1^2 T_2^2 T_3^2 = 0,$$

$$c_1 T_1^5 T_2 + c_2 T_2^5 T_3 + c_3 T_3^5 T_1 - 3\lambda T_1^2 T_2^2 T_3^2 = 0,$$

$$c_1 T_1^4 T_2^2 + c_2 T_2^4 T_3^2 + c_3 T_3^4 T_1^2 - 3\lambda T_1^2 T_2^2 T_3^2 = 0,$$

$$c_1 T_1^3 T_2^2 T_3 + c_2 T_2^3 T_3^2 T_1 + c_3 T_3^3 T_1^2 T_2 - 3\lambda T_1^2 T_2^2 T_3^2 = 0,$$

with $c_i \in k$, $c = c_1 c_2 c_3 \neq 0$ and parameter λ. We suppose that k is of characteristic $p > 3$. We only consider those values of λ such that the corresponding surfaces X_λ are smooth. Let $\hat{c_i} \in W$ be a lifting of c_i for each i and let

$$\hat{c} = \hat{c_1}\hat{c_2}\hat{c_3}.$$

Also take a lifting $\hat{\lambda} \in W$ of λ. Put

$$l(\tau) = \sum_{m=0}^{\infty} b(m) \frac{\tau^{2m+1}}{2m+1},$$

where

$$b(m) = (-3\hat{\lambda})^m {}_3F_2 \left(\begin{array}{c} \frac{-m}{3}, \frac{-m+1}{3}, \frac{-m+2}{3} \\ 1, 1 \end{array} ; \hat{c}\hat{\lambda}^{-3} \right).$$

Define

$$G_{\hat{\lambda}}(\tau_1, \tau_2) = l^{-1}(l(\tau_1) + l(\tau_2)).$$

Then $G_{\hat{\lambda}}$ is a formal group (defined over W) whose reduction G_λ to k realizes $\hat{\mathrm{Br}}(X_\lambda)$ ([13], Theorem 2).

We have the similar consequences as in (4.5.2). Suppose that $b(p-1) \not\equiv 0 \pmod p$ for some λ (i.e., G_λ is of multiplicative type). Take $\hat{c}, \hat{\lambda} \in W$ to be the Teichmüller lifting of c, λ, respectively. Let $\beta_\lambda \in W$ be the p-adic unit such that

$$b\left(\frac{\mu p^{s+1}-1}{2}\right) \equiv \beta_\lambda \cdot b\left(\frac{\mu p^s - 1}{2}\right)^\sigma \pmod{p^{s+1}}$$

for any integers $\mu, s \geq 0$ (with the convention that $b(m) = 0$ if m is not an integer). Here the upper script σ indicates the action by the absolute Frobenius σ on W. In this case, the polynomials

$${}_3F_2\left(\begin{array}{c} \frac{-m_s}{3}, \frac{-m_s+1}{3}, \frac{-m_s+2}{3} \\ 1, 1 \end{array} ; x\right)$$

with $m_s := \frac{1}{2}(p^s - 1)$, converge p-adically to

$${}_3F_2\left(\begin{array}{c} \frac{1}{6}, \frac{3}{6}, \frac{5}{6} \\ 1, 1 \end{array} ; x\right) = \sum_{r=0}^{\infty} 12^{-3r}\binom{6r}{r}\binom{5r}{r}\binom{4r}{r}x^r.$$

(Notice that this formal power series ${}_3F_2$ has coefficients in $\mathbb{Z}[\frac{1}{6}]$.) Thus we have

$$\begin{aligned} \beta_\lambda &= \lim_{s\to\infty} \frac{a\left(\frac{p^{s+1}-1}{2}\right)}{a\left(\frac{p^s-1}{2}\right)^\sigma} \\ &= (-3\hat{\lambda})^{(p-1)/2} \cdot g(\hat{c}\hat{\lambda}^{-3}). \end{aligned}$$

Here $g(x)$ is the formal power series

$$g(x) = {}_3F_2\left(\begin{array}{c} \frac{1}{6}, \frac{3}{6}, \frac{5}{6} \\ 1, 1 \end{array} ; x\right) \Big/ {}_3F_2\left(\begin{array}{c} \frac{1}{6}, \frac{3}{6}, \frac{5}{6} \\ 1, 1 \end{array} ; x^p\right)$$

with coefficients in $\mathbb{Z}_p$, and it converges p-adically at $\hat{c}\hat{\lambda}^{-3}$.

Remark 4.7. Note that if $c_i \in \mathbb{Q}$ in Examples (4.5.1) and (4.6.1), the K3 surfaces X over $\mathbb{Q}$ have geometric Picard rank $\rho(X_{\bar{\mathbb{Q}}}) = 20$. Thus there is a modular form associated to the Galois representation on the (ℓ-adic) transcendental cycles of each such X. For this modularity property, see [17], especially §4 and Proposition 8.12.

(4.8) Examples of K3 surfaces of finite height > 1

Finally we give two examples of K3 surfaces of finite height greater than 1. We introduce some notations. For a positive integer n, denote

$$[n] = \left\{\frac{i}{n} \mid 1 \leq i \leq n, (i, n) = 1\right\}.$$

If m is another positive integer, $[n]^m$ denotes the multi-index such that each element in $[n]$ repeats m-times.

(4.8.1) Examples: Quasi-diagonal K3 surfaces

Consider the family of deformations X_λ, $\lambda \in \mathbb{P}^1_k$, of the quasi-diagonal K3 surface in $\mathbb{P}^3$

$$X_\lambda : T_1^4 + T_1 T_2^3 + T_3^4 + T_4^4 - 12\lambda T_1 T_2 T_3 T_4 = 0.$$

As before, we only consider those values of λ such that the corresponding surfaces X_λ are smooth. Then similar to (4.5.1), when $\lambda = 0$, the formal Brauer group of X_0 is realized as the reduction to k of the formal group law $G_{\hat{0}}$ over $\mathbb{Z}$ with the logarithm

$$l(\tau) = \sum_{n=0}^{\infty} \binom{12n}{2n}\binom{10n}{3n}\binom{7n}{4n}\frac{\tau^{12n+1}}{12n+1}.$$

Similar to (4.5.2), when $\lambda \neq 0$, let $\hat{\lambda} \in W$ be a lifting of λ. Then the formal Brauer group G_λ of X_λ is realized as the reduction to k of the formal group law $G_{\hat{\lambda}}$ over W with the logarithm

$$l(\tau) = \sum_{m=0}^{\infty} (-12\hat{\lambda})^m A_m(\hat{\lambda}) \frac{\tau^{m+1}}{m+1},$$

where

$$A_m(x) = {}_{12}F_{11}\left(\begin{array}{c} \left\{\frac{-m+i}{12} \mid 0 \leq i \leq 11\right\} \\ [1]^3, [2]^2, [3]^2, [4] \end{array} ; \left(2^{10}3^6 x^{12}\right)^{-1} \right).$$

Notice that $A_m(x)$ are polynomials over $\mathbb{Z}[\frac{1}{6}]$.

Assume that $p \neq 2, 3$. Then for $\lambda \in k$, the formal group G_λ is of height one if and only if $\lambda^{p-1} A_{p-1}(\lambda) \neq 0$. Here we regard $x^{p-1} A_{p-1}(x)$ as a polynomial in x over k. Thus G_λ is generically of height one. Put $x = \left(2^{10}3^6\lambda^{12}\right)^{-1}$. Let

$$V_1(x) = A_{p-1}(\lambda),$$

and

$$V_2(x) = \frac{1}{p}\left(A_{p^2-1}(\lambda) - A_{p-1}(\lambda)^{p+1}\right)$$

regarded as polynomials of x. Then in fact $V_2(x) \in W[x]$. Reducing modulo p, we regard $V_i(x)$ as polynomials over k. Then for a non-zero $\lambda \in k$, the formal group G_λ is of height two if and only if $V_1(x) = 0$ but $V_2(x) \neq 0$ (see [9], Lemma (2.1)).

For example, when $p = 13$, as polynomials over k, we have

$$V_1(x) = 1 + 10x$$

and

$$\begin{aligned} V_2(x) &= 8x(x+2)(x+6)(x+10)(x^2+8x+10) \\ &\quad \times (x^8 + 9x^7 + 5x^6 + 12x^5 + 6x^3 + 3x^2 + 7). \end{aligned}$$

One checks that V_1 does not divide V_2. Thus for those $\lambda \in k$ such that $\left(2^{10}3^6\lambda^{12}\right)^{-1} = 9$, the K3 surface X_λ is of height two.

If m runs through those values $m_s := p^s - 1$, the polynomials A_m converges p-adically to

$$A(x) = {}_6F_5\left(\begin{array}{c}[6],[12]\\ [1]^2,[2],[3]\end{array};\left(2^{10}3^6x^{12}\right)^{-1}\right).$$

Notice that $A(x)$ is a formal power series (in x^{-1}) over $\mathbb{Z}_p$ for all primes $p \neq 2, 3$.

(4.8.2) Examples: Double sextic K3 surfaces of quasi-diagonal type

Consider the double cover X_λ, $\lambda \in \mathbb{P}^1_k$, of $\mathbb{P}^2$ ramified along

$$T_1^6 + T_1T_2^5 + T_3^6 - 15\lambda T_1^2T_2^2T_3^2 = 0.$$

As before, we only consider those values of λ such that the corresponding surfaces X_λ are smooth. Then similar to (4.6.1), when $\lambda = 0$, the formal Brauer group of X_0 is realized as the reduction to k of the formal group law $G_{\hat{0}}$ over $\mathbb{Z}$ with the logarithm

$$l(\tau) = \sum_{n=0}^{\infty}\binom{15n}{5n}\binom{10n}{6n}\frac{\tau^{30n+1}}{30n+1}.$$

Similar to (4.6.2), when $\lambda \neq 0$, let $\hat{\lambda} \in W$ be a lifting of λ. Then the formal Brauer group G_λ of X_λ is realized as the reduction to k of the formal group law $G_{\hat{\lambda}}$ over W with logarithm

$$l(\tau) = \sum_{m=0}^{\infty}(-15\hat{\lambda})^m B_m(\hat{\lambda})\frac{\tau^{2m+1}}{2m+1},$$

where

$$B_m(x) = {}_{15}F_{14}\left(\begin{array}{c}\left\{\frac{-m+i}{15} \mid 0 \le i \le 14\right\}\\ [1]^2,[2]^2,[3],[4],[5],[6]\end{array};\left(4^45^56^6x^{15}\right)^{-1}\right).$$

Notice that $B_m(x)$ are polynomials over $\mathbb{Z}[\frac{1}{30}]$.

Assume that $p > 5$. Let $n = \frac{1}{2}(p-1)$. Then for $\lambda \in k$, the formal group G_λ is of height one if and only if $\lambda^n B_n(\lambda) \neq 0$. Here we regard $x^n B_n(x)$ as a polynomial over k. Thus G_λ is generically of height one. Put $x = \left(4^45^56^6\lambda^{15}\right)^{-1}$. Similar to the previous example, let

$$V_1(x) = B_n(\lambda)$$

and

$$V_2(x) = \frac{1}{p}\left(B_{(p^2-1)/2}(\lambda) - B_n(\lambda)^{p+1}\right).$$

One compute that if $p = 31$, then over k we have

$$V_1(x) = 1 + 20x$$

and

$$\begin{aligned} V_2(x) &= 7x + 2x^2 + \cdots + 24x^{32} \\ &= 24x(x^3 + \cdots)(x^6 + \cdots)(x^{22} + \cdots), \end{aligned}$$

where the last three factors are irreducible polynomials of degree 3, 6, and 22 over $\mathbb{F}_{31}$, respectively. We have checked that V_1 and V_2 are relatively prime to each other. Thus for those $\lambda \in k$ such that $\left(4^4 5^5 6^6 \lambda^{15}\right)^{-1} = 17$, the K3 surface X_λ is of height two.

For an odd prime p, if m runs through those values $m_s := \frac{1}{2}(p^s - 1)$, the polynomials B_m converges p-adically to

$$B(x) = {}_{12}F_{11}\left(\begin{matrix} [10], [30] \\ [1]^2, [2], [3], [4], [5] \end{matrix} ; \left(4^4 5^5 6^6 x^{15}\right)^{-1} \right).$$

Notice that $B(x)$ is a formal power series (in x^{-1}) over $\mathbb{Z}_p$ for all $p > 5$.

Question 4.9. For all prime numbers $p < 150$ in the last two examples, we have checked that for any non-zero $\lambda \in k$, the formal group G_λ is either of height 1 or 2, i.e., the two polynomials V_1 and V_2 over $\mathbb{F}_p$ have no non-trivial common divisor. Is it true that G_λ has height ≤ 2 for each $\lambda \neq 0$ and every possible prime p?

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