

Variation of the Unit Root along the Dwork Family of Calabi-Yau Varieties^{*†}

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Abstract

We study the variation of the unit roots of members of the Dwork families of Calabi-Yau varieties over a finite field by the method of Dwork-Katz and also from the point of view of formal group laws. A p -adic analytic formula for the unit roots away from the Hasse locus is obtained.

1 Introduction

Let us first look at an example and then explain the main results of this paper.

(a) *The Legendre family*

Recall the following classical results (see [1], §6(i) or [6], §8).

Consider the Legendre family of elliptic curves E_λ whose affine part is given by

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

with the parameter $\lambda \neq 0, 1$. Over the complex numbers $\lambda \in \mathbb{C}$, the relative de Rham cohomology H_{dR}^1 of degree 1 of the family is free of rank 2. The Hodge filtration $\text{Fil}^1 \subset H_{dR}^1$ is generated by the differential of the first kind

$$\omega = \frac{dx}{y}.$$

Let ∇ be the Gauss-Manin connection on H_{dR}^1 . Then ω satisfies the associated Picard-Fuchs equation $\nabla(L)\omega = 0$, where

$$L = \lambda(1-\lambda)\frac{d^2}{d\lambda^2} + (1-2\lambda)\frac{d}{d\lambda} - \frac{1}{4}. \quad (1)$$

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Up to a constant, the unique holomorphic solution to the differential equation (1) at $\lambda = 0$ is given by the Gauss hypergeometric series

$$F(\lambda) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda\right). \quad (2)$$

Furthermore, the section

$$u = \lambda(1 - \lambda)F\omega' - \lambda(1 - \lambda)F'\omega \quad (3)$$

of H_{dR}^1 is a global horizontal section with respect to $\nabla(d/d\lambda)$. Here $\omega' = \nabla(d/d\lambda)\omega$ and $F' = dF/d\lambda$ are the derivatives.

Interestingly the series (2) also gives information for the Legendre family over a finite field. More precisely, let p be an odd prime. Let $H(x) = F^{<p}(x)$ be the truncation of $F(x)$ up to degree $(p-1)$. Let $\lambda \in \mathbb{F}_q, \lambda \neq 0, 1$, where $q = p^s$. Then the elliptic curve E_λ is ordinary if and only if $H(\lambda) \neq 0$ in \mathbb{F}_q .

Moreover, let $W(\mathbb{F}_q)$ be the ring of Witt vectors of \mathbb{F}_q . Let $\hat{\lambda} \in W(\mathbb{F}_q)$ be the Teichmüller lifting of λ . Suppose $H(\lambda) \neq 0$. Then the formal power series

$$f(x) = \frac{F(x)}{F(x^p)}$$

converges at $\hat{\lambda}$ as a series over $W(\mathbb{F}_q)$. Write the zeta function of E_λ over \mathbb{F}_q as

$$Z(E_\lambda, T) = \frac{1 - aT + qT^2}{(1 - T)(1 - qT)}.$$

Then $a = \pi + \pi'$ with $\pi\pi' = q$ and

$$\pi = \varepsilon^s f(\hat{\lambda})f(\hat{\lambda}^p) \cdots f(\hat{\lambda}^{p^{s-1}}) \quad (4)$$

with $\varepsilon = (-1)^{(p-1)/2}$. The algebraic integer π is a p -adic unit and is called the *unit root* of the ordinary elliptic curve E_λ .

We remark that the formula (4) can be derived easily by the method in §5, based on formal group laws.

(b) *The Dwork family*

In this paper, we shall generalize the above results to certain higher dimensional cases.

Throughout the paper, we let $n \geq 2$ be an integer. Let \mathcal{A} be a base ring in which $(n+1)$ is invertible. Set

$$X = [X_1 : X_2 : \cdots : X_{n+1}]$$

to be the homogeneous coordinates of the projective space \mathbb{P}^n over \mathcal{A} . We will write $t \in \mathbb{A}^1 \cup \{\infty\}$ as the coordinate of \mathbb{P}^1 .

Definition (cf. [9], §1). The *Dwork family* over \mathcal{A} is the one-parameter family V_t of Calabi-Yau hypersurfaces in \mathbb{P}^n over $t \in \mathbb{P}^1$ defined by the equation $\mathcal{P}_t(X) = 0$, where

$$\mathcal{P}_t(X) = X_1^{n+1} + X_2^{n+1} + \cdots + X_{n+1}^{n+1} - (n+1)tX_1X_2 \cdots X_{n+1}. \quad (5)$$

We also set \mathcal{V} to be the total space of the family in $\mathbb{P}^n \times \mathbb{P}^1$ and $v : \mathcal{V} \rightarrow \mathbb{P}^1$ to be the fiber map.

In this paper, we study the variation along t of the zeta function of V_t in the Dwork family over a finite field $\mathcal{A} = \mathbb{F}_q$. Relevant definitions will be given in §2. As in the case of Legendre family, we show that the zeta function of V_t is closely related to the unique holomorphic solution F of the associated Picard-Fuchs equation of the family. For the Dwork family, it turns out that F is a generalized hypergeometric series. We shall define the Hasse invariant for the Dwork family. Similar to the formula (4), we shall derive a formula for the unit root of V_t in terms of the ratio, f , of F and its Frobenius twist (Theorem 4.3). We derive the formula by two different methods, (I) and (II), which we describe briefly now.

- (I) We follow Katz's crystalline interpretation [6] of Dwork's work on the variation of zeta functions of hypersurfaces [1]. We generalize the construction of the horizontal section (3) to the case of the Dwork family (Corollary 3.5). In this way, we see immediately that the Picard-Fuchs equation enters the picture.
- (II) We study the Frobenius action via formal groups. We determine the formal group associated to $H^{n-1}(V_t, W\mathcal{O})$ of V_t explicitly by writing down a formal group law G_t for it (Proposition 5.2). For this, we will follow the work of Stienstra [12]. From this point of view, the series F appears as a certain p -adic limit of the coefficients of the logarithm of G_t .

Besides the formula for the unit root, each method also provides bonus information of different flavor. Method (I) shows that F_i/F has a p -adic analytic continuation, where F_i is the i -th derivative of F . Method (II) provides a good approximation to the unit root similar to [1], Lemma (3.4)(i). The second method might be viewed as a dual approach of [8]. In op.cit., the author studied the highest slope part while here we look at the slope zero part directly.

We study the relation between the Picard-Fuchs equation and the unit root in a hope that, at least over positive characteristic, one can study the arithmetic of the Dwork family or other families of Calabi-Yau varieties inductively by reducing the weights of the cohomology. For example, in the threefold case ($n = 4$), the Picard-Fuchs equation studied here takes care of the most transcendental part of the middle cohomology H^3 . Then the remaining part can be viewed as a family of abelian varieties by the construction of intermediate Jacobians. One might combine these two pieces to obtain a better understanding of the whole H^3 . Also the methods developed in this paper should be

extendable to families of Calabi-Yau varieties of generalized hypergeometric type (e.g., complete intersections in weighted projective spaces).

The original motivation of studying the associated formal group for the Dwork family was to see if one can get a geometric interpretation of the congruence [1], Lemma (3.4)(i), and eventually the congruences in §1, Corollary 2 therein. If this is the case, one might be able to prove the similar congruences for different families of Calabi-Yau varieties of non-hypergeometric type. Numerical computation suggests that the *Apéry numbers* listed in [13], Table 7 satisfy the same type of congruences in loc.cit. Unfortunately, Method (II) gives rise to a family of hypergeometric functions different from the truncations of F , and the congruences for Apéry numbers remain as an open question.

Recently there are several papers dealing with the Dwork family. In [3], the authors compute the Zariski closure of the monodromy group in characteristic 0 of this family and apply it to the study of the Sato-Tate conjecture. The papers [9] and [11] also investigate the family in characteristic p via ℓ -adic Fourier transforms. The moment zeta functions of the Dwork family are computed in [11]. In [10], the zeta functions for more general monomial deformations of Fermat type hypersurfaces in weighted projective spaces are studied via Dwork's deformation theory.

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2 Preliminary

In this section, we give relevant definitions that are used in this paper.

Fix an integer $n \geq 2$. Recall that \mathcal{A} is the base ring in which $(n+1)$ is invertible. The map $v : \mathcal{V} \rightarrow \mathbb{P}^1$ indicates the fiber map of the Dwork family (see (5))

$$V_t : \mathcal{P}_t(X) = 0.$$

Let

$$\mu_{n+1} = \text{Spec } \mathcal{A}[x]/(x^{n+1} - 1)$$

be the group scheme of the $(n+1)$ -st roots of unity over \mathcal{A} . Note that over the subscheme $\mathcal{T} = \mathbb{A}^1 \setminus \mu_{n+1}$ in \mathbb{P}^1 , the map v is smooth. Let $\mathcal{H} = \mu_{n+1}^{n+1}/\mu_{n+1}$, where the quotient is via the diagonal embedding. Let \mathcal{H}_0 be the subgroup of \mathcal{H} defined by

$$\mathcal{H}_0 = \{ \underline{\zeta} = (\zeta_1, \dots, \zeta_{n+1}) \in \mathcal{H} \mid \zeta_1 \cdots \zeta_{n+1} = 1 \}.$$

Then \mathcal{H}_0 acts on each fiber V_t of the family by

$$\underline{\zeta}(x_1, x_2, \dots, x_{n+1}) = (\zeta_1 x_1, \zeta_2 x_2, \dots, \zeta_{n+1} x_{n+1})$$

for $x = (x_1, \dots, x_{n+1}) \in V_t$. We are mainly interested in the relative cohomology of v of degree $(n-1)$ fixed by \mathcal{H}_0 . We discuss the details in the following two cases.

(a) *Over \mathbb{C}*

Suppose $\mathcal{A} = \mathbb{C}$. Let \mathcal{M}_{dR} be the fixed part by \mathcal{H}_0 of the relative de Rham cohomology $R^{n-1}v_*\Omega_{V/\mathcal{T}}^\bullet$ of degree $(n-1)$ over \mathcal{T} . Here $\Omega_{V/\mathcal{T}}^\bullet$ is the complex of relative differential forms. Then the sheaf \mathcal{M}_{dR} is locally free of rank n . Denote by Fil^\bullet the Hodge filtration of \mathcal{M}_{dR} . At each point, \mathcal{M}_{dR} has Hodge numbers

$$h^{i, n-1-i} := \dim_{\mathbb{C}} \text{Fil}^i / \text{Fil}^{i+1} = 1 \quad (6)$$

for all $0 \leq i \leq n-1$ ([3], §1). Let ∇ be the Gauss-Manin connection. Then \mathcal{M}_{dR} is stable under ∇ .

Let Ω be the differential n -form on \mathbb{P}^n defined by

$$\Omega = \sum_{i=1}^{n+1} (-1)^i X_i dX_1 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_{n+1}.$$

Here $\widehat{dX_i}$ means the deletion of the i -th component dX_i . Let

$$\xi = \text{Res}_{V_t}(\Omega/\mathcal{P}_t)$$

be the cohomology class in the top forms

$$H^0(V_t/\mathbb{C}, \Omega^{n-1}) \subset H_{dR}^{n-1}(V_t/\mathbb{C})$$

given by the residue of the meromorphic differential n -form Ω/\mathcal{P}_t . Let

$$\xi_i = \left(\nabla \left(\frac{d}{dt} \right) \right)^i \xi$$

be the i -th derivative of ξ . It is shown (loc.cit) that

$$\xi_i \in \text{Fil}^{n-1-i} - \text{Fil}^{n-i}$$

and the set $\{\xi_i\}_{0 \leq i < n}$ generates \mathcal{M}_{dR} . Finally define a new section η of \mathcal{M}_{dR} by

$$\eta = t \cdot \xi. \quad (7)$$

Proposition 2.1 *In \mathcal{M}_{dR} , we have*

(1) ξ satisfies the Picard-Fuchs equation $\nabla(L_t)\xi = 0$, where

$$L_t = \prod_{i=1}^n (\delta + 1 - i) - t^{n+1} (\delta + 1)^n \quad (8)$$

with $\delta = \delta_t = t \frac{d}{dt}$.

(2) η satisfies the differential equation $\nabla(L_\lambda)\eta = 0$, where

$$L_\lambda = \theta^n - \lambda \prod_{i=1}^n \left(\theta + \frac{i}{n+1} \right) \quad (9)$$

with $\lambda = t^{-(n+1)}$ and $\theta = \theta_\lambda = \lambda \frac{d}{d\lambda}$.

Proof. We prove that (8) and (9) are indeed the Picard-Fuchs equations for ξ and η , respectively, as a follow-up of [3], §1. To shorten the notation, we simply write $D\omega$ to indicate $\nabla(D)\omega$ for any derivative D and de Rham cohomology class ω .

We employ the same notations as in [3]. Let $\omega_1 = \eta = t \cdot \xi$ and define ω_r inductively by

$$\omega_{r+1} = \delta\omega_r - \omega_r.$$

For all $0 \leq i < j$, define the constants $A_{i,j}$ by

$$T^r = \sum_{i=0}^r A_{i,r+1} (T-1)(T-2) \cdots (T-r+i)$$

as polynomials in T for all $r \geq 0$. Then in the first half of §1 in op.cit. (formula in page 9), we have, as cohomology classes,

$$\omega_{n+1} = t^{n+1} (A_{0,n+1}\omega_{n+1} + A_{1,n+1}\omega_n + \cdots + A_{n,n+1}\omega_1). \quad (10)$$

Inductively, we can derive

$$\omega_{r+1} = (\delta - 1)(\delta - 2) \cdots (\delta - r)\omega_1, \quad (11)$$

and

$$\delta^r \omega_1 = t(\delta + 1)^r \xi. \quad (12)$$

Plugging (11) into (10), we get

$$\begin{aligned} \left(\prod_{i=1}^n (\delta - i) \right) \omega_1 &= \left(t^{n+1} \sum_{i=0}^n A_{i,n+1} (\delta - 1)(\delta - 2) \cdots (\delta - n + i) \right) \omega_1 \\ &= t^{n+1} \delta^n \omega_1. \end{aligned} \quad (13)$$

Equations (12) and (13) imply

$$t \prod_{i=1}^n (\delta + 1 - i) \xi = t^{n+2} (\delta + 1)^n \xi,$$

which gives the claimed equation (8).

On the other hand, set $\lambda = t^{-(n+1)}$ and $\theta = \lambda \frac{d}{d\lambda}$. Then

$$\delta = -(n+1)\theta.$$

Putting it into (13), we obtain

$$\prod_{i=1}^n (-(n+1)\theta - i)\eta = \lambda^{-1} (-(n+1)\theta)^n \eta,$$

which is equivalent to (9). □

We remark that the discussion here is also valid in the case of any field that can be embedded into \mathbb{C} .

(b) *Over positive characteristic*

For relevant descriptions of crystals over a smooth base, see [6], §§1, 5, 7 and [7], §§2.1, 2.4.

Let $\mathcal{A} = k$ be a field of characteristic $p > 0$ with $(p, n+1) = 1$. Let $W = W(k)$ be the ring of Witt vectors of k . For $t \in k, t^{n+1} \neq 1$, the crystalline cohomology $H_{cris}^{n-1}(V_t/W)$ is a free W -module equipped with an absolute Frobenius action ϕ . We will simply call the Newton polygon of V_t to mean the Newton polygon of $H_{cris}^{n-1}(V_t/W)$, and similarly for the Hodge polygon (see [7], §§1.2 and 1.3).

Definition. We say that V_t is *ordinary* if the Newton polygon coincides with the Hodge polygon of V_t (cf. [5], §§1.1, 1.3).

Theorem 2.2 *Let k be a field of characteristic $p > 0$ with $(p, n+1) = 1$. Then the Dwork family defined by equation (5) over k is generically ordinary.*

Proof. Let $k[[t^{-1}]]$ be the localization of the parameter space \mathbb{P}^1 at $t = \infty$. Let $\mathcal{T}' = \text{Spec } k[[t^{-1}]]$ and \mathcal{V}' be the restriction of the family \mathcal{V} to \mathcal{T}' . The special fiber V_∞ is the union of coordinate hyperplanes. Each arbitrary intersection among them is isomorphic to some projective space and is obviously ordinary. Thus the generic fiber $\tilde{\mathcal{V}}'$ of \mathcal{V}' is ordinary ([5], Proposition 1.10). □

Let $\hat{t} \in W$ be a lifting of $t \in k$. Then $H_{dR}^{n-1}(V_{\hat{t}}/W)$ is canonically isomorphic to $H_{cris}^{n-1}(V_t/W)$ and the identification is compatible with the \mathcal{H}_0 -action. Let \mathcal{M}_{cris} be the fixed part of $H_{cris}^{n-1}(V_t/W)$ by \mathcal{H}_0 . Then \mathcal{M}_{cris} is a direct summand of $H_{cris}^{n-1}(V_t/W)$, and similarly for the fixed part \mathcal{M}_{dR} of $H_{dR}^{n-1}(V_{\hat{t}}/W)$ by \mathcal{H}_0 . Since the Newton polygon is on or above the Hodge polygon of V_t , and by a glance at the Hodge polygon of \mathcal{M}_{dR} described in (6), we see that the first slope of the Newton polygon of V_t must be 0, 1/2, or ≥ 1 .

Assume now that k is perfect. Let \mathfrak{Art}_k be the category of Artinian local k -algebras, and \mathfrak{AG} the category of abstract abelian groups. The Artin-Mazur functor

$$G_t : \mathfrak{Art}_k \rightarrow \mathfrak{AG}$$

is defined by

$$G_t(R, \mathfrak{m}) = \text{Ker} \left\{ H_{et}^{n-1} \left(X \otimes_k R, \hat{\mathbb{G}}_m \right) \rightarrow H_{et}^{n-1} \left(X \otimes_k R/\mathfrak{m}, \hat{\mathbb{G}}_m \right) \right\},$$

for (R, \mathfrak{m}) an object in $\mathfrak{A}tt_k$. The functor G_t is pro-representable by a one-dimensional commutative formal group ([12], Theorem 1). We call G_t the *formal group associated to V_t* . The (covariant) Cartier module (of p -typical curves) of G_t is canonically isomorphic to $H^{n-1}(V_t, W\mathcal{O})$ as a $W[\phi]$ -module ([4], Remarque II.2.15). By the description of the Newton polygon in the last paragraph, G_t is of height 1, 2 or ∞ . Notice that the formal group G_t can be defined over a more general base ring \mathcal{A} (see [12], Theorem 1).

Finally suppose $k = \mathbb{F}_q$ is a finite field of q elements. Suppose $t \in \mathbb{F}_q, t^{n+1} \neq 1$. If the first slope of the Newton polygon of V_t is zero, there is a unique p -adic unit root of the geometric Frobenius endomorphism acting on $H_{cris}^{n-1}(V_t/W)$. We will call this the *unit root* of V_t .

3 Existence of a global horizontal section

Here we explicitly construct a global horizontal section for the Dwork family with respect to the Gauss-Manin connection over characteristic zero.

Lemma 3.1 *For any two positive integers k, m with $(k-1) < m$, we have*

$$\sum_{r=0}^m (-1)^r \binom{m-k+r}{k-1} \binom{m}{r} = 0.$$

Proof. Consider the function $a(x) = x^{m-k}(1+x)^m$. Then

$$(k-1)! \sum_{r=0}^m (-1)^r \binom{m-k+r}{k-1} \binom{m}{r} = \pm \frac{d^{k-1}}{dx^{k-1}} a(-1) = 0$$

for $(k-1) < m$. □

Lemma 3.2 *Consider functions $b_i^{(m)} = b_i^{(m)}(a_1, \dots, a_m), 0 \leq i \leq 2m$, of m variables a_1, \dots, a_m defined by*

$$b(x) = \prod_{i=1}^m (x+a_i)(x+1-a_i) = \sum_{i=0}^{2m} b_i^{(m)} x^{2m-i}.$$

Then for $1 \leq k \leq m$, we have

$$\sum_{i=0}^{2m-2k+1} (-1)^i \binom{k-1+i}{k-1} b_{2m-2k+1-i}^{(m)} = 0.$$

Proof. We prove this by deformation of a_i and by induction on m . It is easy to establish the equality when $m = 1$. Then notice first that

$$\begin{aligned}\frac{\partial b(x)}{\partial a_i} &= \frac{b(x)}{x + a_i} - \frac{b(x)}{x + 1 - a_i} \\ &= (1 - 2a_i) \frac{b(x)}{(x + a_i)(x + 1 - a_i)}.\end{aligned}$$

Comparing the coefficients on both sides, we have

$$\frac{\partial}{\partial a_i} b_k^{(m)}(a_1, \dots, a_m) = (1 - 2a_i) b_{k-2}^{(m-1)}(a_1, \dots, \hat{a}_i, \dots, a_m).$$

Here \hat{a}_i means the deletion of the i -th component a_i . Thus by induction,

$$\begin{aligned}\frac{\partial}{\partial a_i} \sum_{i=0}^{2m-2k+1} (-1)^i \binom{k-1+i}{k-1} b_{2m-2k+1-i}^{(m)} \\ &= (1 - 2a_i) \sum_{i=0}^{2m-2k-1} (-1)^i \binom{k-1+i}{k-1} b_{2m-2k-1-i}^{(m-1)} \\ &= 0.\end{aligned}$$

Therefore $\sum (-1)^i \binom{k-1+i}{k-1} b_{2m-2k+1-i}^{(m)}$ is a constant.

Secondly for $a_1 = \dots = a_m = 0$, we have $b_i^{(m)} = \binom{m}{i}$ and then

$$\begin{aligned}\sum_{i=0}^{2m-2k+1} (-1)^i \binom{k-1+i}{k-1} b_{2m-2k+1-i}^{(m)} &= \sum_{i=0}^{2m-2k+1} (-1)^i \binom{k-1+i}{k-1} \binom{m}{2m-2k+1-i} \\ &= \sum_{i=m-2k+1}^{2m-2k+1} (-1)^i \binom{k-1+i}{k-1} \binom{m}{2m-2k+1-i} \\ &= \sum_{r=0}^m (-1)^r \binom{m-k+r}{k-1} \binom{m}{r} \\ &= 0\end{aligned}$$

by Lemma 3.1 and this completes the proof. \square

Lemma 3.3 Consider functions $\beta_i = \beta_i(a_1, \dots, a_m)$, $0 \leq i \leq 2m + 1$, of variables a_1, \dots, a_m defined by

$$\beta(x) = \left(x + \frac{1}{2}\right) \prod_{i=1}^m (x + a_i)(x + 1 - a_i) = \sum_{i=0}^{2m+1} \beta_i x^{2m+1-i}.$$

Then

$$\sum_{i=0}^{2m} (-1)^i \beta_{2m-i} = 2\beta_{2m+1},$$

and for $1 \leq k \leq m$, we have

$$\sum_{i=0}^{2m-2k+1} (-1)^i \binom{k-1+i}{k-1} \beta_{2m-2k+1-i} = \frac{1}{2} \sum_{i=0}^{2m-2k} (-1)^i \binom{k+i}{k} \beta_{2m-2k-i}.$$

Proof. We define $b_i = b_i^{(m)}(a_1, \dots, a_m)$, $0 \leq i \leq 2m$ as in Lemma 3.2. Let $b_{-1} = b_{2m+1} = 0$. Then

$$\beta_i = b_i + \frac{1}{2} b_{i-1}$$

for $0 \leq i \leq 2m+1$. Thus by Lemma 3.2,

$$\begin{aligned} \sum_{i=0}^{2m} (-1)^i \beta_{2m-i} &= \sum_{i=0}^{2m} (-1)^i \left(b_{2m-i} + \frac{1}{2} b_{2m-i-1} \right) \\ &= b_{2m} - \frac{1}{2} \sum_{i=0}^{2m-1} (-1)^i b_{2m-1-i} \\ &= b_{2m} \\ &= 2\beta_{2m+1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\sum_{i=0}^{2m-2k+1} (-1)^i \binom{k-1+i}{k-1} \beta_{2m-2k+1-i} \\ &= \sum_{i=0}^{2m-2k+1} (-1)^i \binom{k-1+i}{k-1} \left(b_{2m-2k+1-i} + \frac{1}{2} b_{2m-2k-i} \right) \\ &= \sum_{i=0}^{2m-2k+1} (-1)^i \binom{k-1+i}{k-1} b_{2m-2k+1-i} + \frac{1}{2} \sum_{i=0}^{2m-2k+1} (-1)^i \binom{k-1+i}{k-1} b_{2m-2k-i} \\ &= \frac{1}{2} \sum_{i=0}^{2m-2k} (-1)^i \binom{k-1+i}{k-1} b_{2m-2k-i} \end{aligned}$$

by Lemma 3.2, and since $b_{-1} = 0$. Similarly,

$$\sum_{i=0}^{2m-2k} (-1)^i \binom{k+i}{k} \beta_{2m-2k-i} = \sum_{i=0}^{2m-2k} (-1)^i \binom{k+i}{k} \left(b_{2m-2k-i} + \frac{1}{2} b_{2m-2k-1-i} \right),$$

and

$$\sum_{i=0}^{2m-2k} (-1)^i \binom{k+i}{k} b_{2m-2k-1-i} = 0.$$

Thus we have the desired equality. \square

Theorem 3.4 *Let A be a ring. Let B and M be two $A[x]/A$ -differential modules, where x is a variable. Fix an $A[x]/A$ -differential D and assume $e \in A[x]$ satisfying $De = e$. Suppose $\{a_i\} \subset A$ is stable under the transformation $a_i \mapsto 1 - a_i$. Suppose that $g \in B$ and $\eta \in M$ satisfy the differential equation $Lv = 0$, where*

$$L = D^n - e \prod_{i=1}^n (D + a_i) = D^n - e \sum_{i=0}^n b_i D^{n-i}. \quad (14)$$

Write $n = 2m - \varepsilon$, where $m \in \mathbb{Z}$, $\varepsilon = 0$ or 1 . Set $g^{(i)} = D^i g$, and $\eta^{(j)} = D^j \eta$. Let

$$c_{ij} = \sum_{r=0}^j (-1)^r \binom{i+r-1}{i-1} b_{j-r}.$$

Then the element $u \in B \otimes_{A[x]} M$ defined by

$$\begin{aligned} u = & (1-e) \sum_{i=0}^{n-1} (-1)^i g^{(i)} \eta^{(n-1-i)} \\ & + e \sum_{i=1}^{m-1} \sum_{j=1}^{n-2i} (-1)^i c_{ij} \left[g^{(i-1)} \eta^{(n-i-j)} - (-1)^\varepsilon g^{(n-i-j)} \eta^{(i-1)} \right] \\ & + \varepsilon e \sum_{i=1}^{m-1} (-1)^i c_{i,n+1-2i} g^{(i-1)} \eta^{(i-1)} \end{aligned} \quad (15)$$

is a horizontal section with respect to D .

Proof. Notice that by an easy calculation, we have

$$c_{i+1,0} = b_0 = 1,$$

$$c_{1,j} + c_{1,j+1} = b_{j+1},$$

and

$$c_{i+2,j} + c_{i+2,j+1} = c_{i+1,j+1} \quad (16)$$

for all $i, j \geq 0$. To simplify the notation, we let

$$(g_i, \eta_j) := g^{(i)} \eta^{(j)} - (-1)^\varepsilon g^{(j)} \eta^{(i)}.$$

We have

$$\begin{aligned} Du = & (1-e)(g, \eta_n) - e \sum_{i=0}^{m-1} (-1)^i (g_i, \eta_{n-1-i}) \\ & + e \sum_{i=1}^{m-1} \sum_{j=1}^{n-2i} (-1)^i c_{ij} \left((g_{i-1}, \eta_{n+1-i-j}) + (g_i, \eta_{n-i-j}) + (g_{i-1}, \eta_{n-i-j}) \right) \\ & + \varepsilon e \sum_{i=1}^{m-1} (-1)^i c_{i,n+1-2i} \left((g_{i-1}, \eta_i) + \frac{1}{2} (g_{i-1}, \eta_{i-1}) \right). \end{aligned}$$

We now distinguish two cases.

(a) Suppose n is even, i.e., $\varepsilon = 0$. Letting $k = i + 1$ in Lemma 3.2, we get

$$c_{i+1, n-1-2i} = 0$$

for $0 \leq i \leq m - 1$. To simplify the notation, we let

$$[g_i, \eta_j] := g^{(i)}\eta^{(j)} - g^{(j)}\eta^{(i)}.$$

Collecting all terms of the form $[g, \eta_j]$ in Du , we have

$$\begin{aligned} & (1 - e)[g, \eta_n] - e \left\{ (1 + c_{1,1})[g, \eta_{n-1}] + \sum_{j=2}^{n-2} (c_{1,j-1} + c_{1,j})[g, \eta_{n-j}] + c_{1,n-2}[g, \eta_1] \right\} \\ = & (1 - e)[g, \eta_n] - e \sum_{j=1}^{n-1} (c_{1,j-1} + c_{1,j})[g, \eta_{n-j}] \\ = & (1 - e)[g, \eta_n] - e \sum_{j=1}^{n-1} b_j [g, \eta_{n-j}] \\ = & g[(L + eb_n)\eta] - \eta[(L + eb_n)g] \\ = & 0. \end{aligned}$$

The last equality follows since $Lg = L\eta = 0$. On the other hand, the coefficient of $[g_i, \eta_j]$ for $0 < i < j$ in Du is given by

$$\begin{aligned} & (-1)^{i+1} e \times \begin{cases} (1 + c_{i+1,1} - c_{i,1}) & \text{if } j = n - 1 - i \\ (c_{i+1, n-1-i-j} + c_{i+1, n-i-j} - c_{i, n-i-j}) & \text{if } n - 2 - i \geq j \geq 2 + i \\ (c_{i+1, n-2-2i} - c_{i, n-1-2i}) & \text{if } j = 1 + i \end{cases} \\ = & (-1)^{i+1} e (c_{i+1, j-1} + c_{i+1, j} - c_{i, j}) \\ = & 0. \end{aligned}$$

The last equality follows from (16). Thus $Du = 0$ in this case.

(b) Suppose n is odd, i.e., $\varepsilon = 1$. Letting $k = i$ in Lemma 3.3, we get

$$c_{1, n-1} = 2b_n$$

and

$$c_{i, n-2i} = \frac{1}{2} c_{i+1, n-1-2i}$$

for $1 \leq i \leq m - 1$. To simplify the notation, we let

$$\{g_i, \eta_j\} := g^{(i)}\eta^{(j)} + g^{(j)}\eta^{(i)}.$$

Collecting all terms of the form $\{g, \eta_j\}$ in Du , we have

$$\begin{aligned}
& (1-e)\{g, \eta_n\} - e \left[(1+c_{1,1})\{g, \eta_{n-1}\} + \sum_{j=2}^{n-1} (c_{1,j-1} + c_{1,j})\{g, \eta_{n-j}\} + c_{1,n-1}g\eta \right] \\
= & (1-e)\{g, \eta_n\} - e \left[\sum_{j=1}^{n-1} (c_{1,j-1} + c_{1,j})\{g, \eta_{n-j}\} + b_n\{g, \eta\} \right] \\
= & (1-e)\{g, \eta_n\} - e \sum_{j=1}^n b_j\{g, \eta_{n-j}\} \\
= & gL\eta + \eta Lg \\
= & 0.
\end{aligned}$$

On the other hand, the coefficient of $\{g_i, \eta_j\}$ for $0 < i \leq j$ in Du is given by

$$\begin{aligned}
& (-1)^{i+1}e \times \begin{cases} (1+c_{i+1,1}-c_{i,1}) & \text{if } j = n-1-i \\ (c_{i+1,n-1-i-j} + c_{i+1,n-i-j} - c_{i,n-i-j}) & \text{if } n-2-i \geq j \geq 1+i \\ (\frac{1}{2}c_{i+1,n-1-2i} - c_{i,n-2i}) & \text{if } j = i \end{cases} \\
= & (-1)^{i+1}e \times \begin{cases} (c_{i+1,n-1-i-j} + c_{i+1,n-i-j} - c_{i,n-i-j}) & \text{if } n-1-i \geq j \geq 1+i \\ (\frac{1}{2}c_{i+1,n-1-2i} - c_{i,n-2i}) & \text{if } j = i \end{cases} \\
= & 0.
\end{aligned}$$

Thus $Du = 0$. □

Remark. If we write the element u in Theorem 3.4 as

$$u = \sum_{i=0}^{n-1} C_i \eta^{(n-1-i)},$$

then $C_0 = (1-e)g$ and C_i is an $A[e]$ -linear combination of $\{g, g^{(1)}, \dots, g^{(i)}\}$.

Now we go back to the Dwork family over \mathbb{C} . The differential equation (9) has a unique power series solution $F(\lambda) \in \mathbb{C}[[\lambda]]$ with constant term 1, which is holomorphic near $\lambda = 0$. We know explicitly that $F(\lambda)$ is given by a hypergeometric series:

$$\begin{aligned}
F(\lambda) &= {}_nF_{n-1} \left(\begin{matrix} \frac{1}{n+1}, \frac{2}{n+1}, \dots, \frac{n}{n+1} \\ 1, 1, \dots, 1 \end{matrix} ; \lambda \right) \\
&= \sum_{r=0}^{\infty} \frac{\left(\frac{1}{n+1}\right)_r \left(\frac{2}{n+1}\right)_r \cdots \left(\frac{n}{n+1}\right)_r}{(r!)^n} \lambda^r,
\end{aligned} \tag{17}$$

where $(a)_0 = 1$ and $(a)_r = a(a+1)\cdots(a+r-1)$ for $r > 0$ is the *Pochhammer symbol*.

Corollary 3.5 *Consider the Dwork family over $\mathcal{A} = \mathbb{C}$. Let $g = F(\lambda)$ be the hypergeometric series (17). Let η be the de Rham cohomology class given by (7). Then up to a constant, the element u constructed in Theorem 3.4 is the unique horizontal section with respect to $\nabla(\frac{d}{d\lambda})$ near $\lambda = 0$.*

Proof. The Picard-Fuchs equation (9) satisfies the condition in Theorem 3.4 with $A = \mathbb{C}$, $B = \mathbb{C}[[\lambda]]$, $M = \mathcal{M}_{dR}$, $D = \lambda\nabla(d/d\lambda)$, $e = \lambda$, $g = F(\lambda)$, and η as defined in (7). Hence u is horizontal with respect to $\lambda\nabla(d/d\lambda)$. Thus it is horizontal with respect to $\nabla(\lambda/d\lambda)$. The uniqueness follows by the computation of the local monodromy near $\lambda = 0$ (see [3], Corollary 1.7). \square

4 General properties and the method of Dwork and Katz

In this section, we mainly consider the Dwork family defined over the base $\mathcal{A} = \mathbb{F}_q$. We define the Hasse invariant of the Dwork family over characteristic p . We derive the formula for the unit root when the first slope of the Newton polygon of V_t is zero via the crystalline approach.

(a) *The Hasse invariant*

In the rest of this paper, let $F(x)$ denote the hypergeometric series (17)

$$\begin{aligned} F(x) &= {}_nF_{n-1} \left(\begin{matrix} \frac{1}{n+1}, \frac{2}{n+1}, \dots, \frac{n}{n+1} \\ 1, 1, \dots, 1 \end{matrix} ; x \right) \\ &= \sum_{r=0}^{\infty} \frac{\left(\frac{1}{n+1}\right)_r \left(\frac{2}{n+1}\right)_r \cdots \left(\frac{n}{n+1}\right)_r}{(r!)^n} x^r \\ &= \sum_{r=0}^{\infty} \left(\prod_{i=1}^{n+1} \binom{ri}{i} \right) \left(\frac{x}{(n+1)^{n+1}} \right)^r. \end{aligned}$$

The last expression shows that $F(x)$ is a formal power series with coefficients in $\mathbb{Z}[\frac{1}{n+1}]$. For any positive integer s , let $F^{<s}(x)$ be the truncated polynomial of the series $F(x)$ up to degree $(s-1)$. We will freely regard $F^{<s}(x)$ as a polynomial in any $\mathbb{Z}[\frac{1}{n+1}]$ -algebra. Similarly for the formal power series $F(x)$.

Definition. Fix a prime p with $(p, n+1) = 1$. The function $H(x) = F^{<p}(x)$, regarded as a polynomial over \mathbb{F}_p , is called the *Hasse invariant* of the Dwork family V_t over characteristic p .

Let A_m be the coefficient of $(X_1 \cdots X_{n+1})^m$ in $\mathcal{P}_t(X)^m$. Thus A_m is a polynomial in t of degree m .

Lemma 4.1 *We have*

- (i) *If $q = p^r$, then $A_{q-1} \equiv (A_{p-1})^{(q-1)/(p-1)} \pmod{p}$.*
- (ii) *Let $\lambda = t^{-(n+1)}$. As polynomials in t , we have $A_{p-1} \equiv t^{p-1}H(\lambda) \pmod{p}$.*

Proof. (i) We have

$$\begin{aligned} \mathcal{P}_t(X)^{q-1} &= \left(\mathcal{P}_t(X)^{1+p+\dots+p^{r-1}} \right)^{p-1} \\ &\equiv \left(\mathcal{P}_t(X) \mathcal{P}_{t^p}(X^p) \dots \mathcal{P}_{t^{p^{r-1}}}(X^{p^{r-1}}) \right)^{p-1} \pmod{p}. \end{aligned}$$

Notice that the coefficient B_i of $(X_1 \dots X_{n+1})^{p^i(p-1)}$ in $\mathcal{P}_{t^{p^i}}(X^{p^i})$ is congruent to $A_{p-1}^{p^i} \pmod{p}$. By an inspection of the terms in the product, we have, in $\mathbb{F}_p[t]$,

$$\begin{aligned} A_{q-1} &= B_0 B_1 \dots B_{r-1} \\ &= A_{p-1} A_{p-1}^p \dots A_{p-1}^{p^{r-1}}. \end{aligned}$$

(ii) We have

$$\begin{aligned} A_{p-1} &= \sum_{i \geq 0} (-(n+1)t)^{p-1-(n+1)i} \binom{p-1}{i} \binom{p-1-i}{i} \dots \binom{p-1-ni}{i} \\ &= (-(n+1)t)^{p-1} \sum_{i \geq 0} \left((-(n+1))^{-(n+1)i} \prod_{k=0}^n \binom{p-1-ki}{i} \right) \lambda^i \\ &\equiv t^{p-1} \sum_{i=0}^{\lfloor \frac{p-1}{n+1} \rfloor} \left(\prod_{k=0}^{(n+1)i} \frac{k}{n+1} \right) \frac{\lambda^i}{(i!)^{n+1}} \pmod{p} \\ &\equiv t^{p-1} \sum_{i=0}^{p-1} \left(\prod_{k=0}^{(n+1)i} \frac{k}{n+1} \right) \frac{\lambda^i}{(i!)^{n+1}} \pmod{p} \\ &= t^{p-1} H(\lambda). \end{aligned}$$

□

Theorem 4.2 *Let \mathbb{F}_q be a finite field of q elements with q a power of p and $(p, n+1) = 1$. Let $H(x) = F^{<p}(x)$ be the Hasse invariant of the Dwork family. Let $t \in \mathbb{F}_q, t \neq 0, t^{n+1} \neq 1$. Let $\lambda = t^{-(n+1)}$. Then the first slope of the Newton polygon of V_t is zero if and only if $H(\lambda) \neq 0$. In this case, if π_t is the unit root of V_t , then*

$$\pi_t \equiv H(\lambda)^{(q-1)/(p-1)} \pmod{p}.$$

Proof. (a) The method here is similar to the case of counting points of Legendre family of elliptic curves. Let

$$N_t = \#\{x \in \mathbb{P}^n(\mathbb{F}_q) \mid x \in V_t(\mathbb{F}_q)\}, \quad \text{and} \quad N'_t = \#\{x \in \mathbb{F}_q^{n+1} \mid \mathcal{P}_t(x) = 0\}.$$

Then

$$N_t = \frac{N'_t - 1}{q - 1} \equiv 1 - N'_t \pmod{q}.$$

Consider the zeta function of V_t

$$\begin{aligned} Z(V_t, T) &= \exp(N_t T + \mathcal{O}(T^2)) \\ &\equiv 1 + N_t T \pmod{T^2}. \end{aligned}$$

Let

$$\det(1 - T \text{Frob}^* | H_{\text{ét}}^{n-1}(V_t \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell)) = 1 - aT + \mathcal{O}(T^2)$$

be the reciprocal characteristic polynomial of the geometric Frobenius Frob^* on the middle cohomology of V_t ($\ell \neq p$). Then by the Weil conjecture

$$\begin{aligned} Z(V_t, T) &= \frac{(1 - aT + \mathcal{O}(T^2))^{(-1)^n}}{(1 - T)(1 - qT^2) \cdots (1 - q^n T^{2n})} \\ &\equiv (1 - (-1)^n aT)(1 + T) \pmod{T^2} \\ &\equiv 1 + (1 - (-1)^n a)T \pmod{T^2}. \end{aligned}$$

Thus $a \equiv (-1)^n N'_t \pmod{q}$, and the theorem is equivalent to say that for $\lambda \neq 0, 1$, the congruence $N'_t \equiv 0 \pmod{p}$ holds if and only if $H(\lambda) = 0$ in \mathbb{F}_q .

By Warning's method,

$$N'_t = \sum_{x \in \mathbb{F}_q^{n+1}} (1 - \mathcal{P}_t(x)^{q-1}).$$

Notice that

$$\sum_{z \in \mathbb{F}_q} z^r \equiv \begin{cases} -1 & \text{if } (q-1) | r \\ 0 & \text{otherwise.} \end{cases}$$

Thus by an inspection of the terms in the expansion of \mathcal{P}^{q-1} ,

$$(-1)^n \sum_{x \in \mathbb{F}_q^{n+1}} (1 - \mathcal{P}_t(x)^{q-1}) \equiv \text{the coeff. } A_{q-1} \text{ of } (X_1 \cdots X_{n+1})^{q-1} \text{ in } \mathcal{P}^{q-1} \pmod{p}.$$

Therefore by Lemma 4.1, the congruence $N'_t \equiv 0 \pmod{p}$ holds if and only if

$$0 = (t^{p-1} H(\lambda))^{(q-1)/(p-1)} = H(\lambda)^{(q-1)/(p-1)},$$

since $t^{q-1} = 1$.

(b) Suppose now $H(\lambda) \neq 0$, then we have

$$\pi_t \equiv a \equiv A_{q-1} \equiv H(\lambda)^{(q-1)/(p-1)} \pmod{p}$$

by the above calculation. □

Remark. The proof of the theorem also shows that over \mathbb{F}_q , the first slope of the variety V_0 in the family at $t = 0$ is zero if and only if $H(x)$ is exactly of degree $(p-1)/(n+1)$ as an element in $\mathbb{F}_q[x]$. This condition is equivalent to the congruence $p \equiv 1 \pmod{n+1}$. The Newton polygon of V_0 can be determined by looking at the splitting type of the prime p in the cyclotomic field $\mathbb{Q}(\mu_{n+1})$ (see [2], Proposition 3.8).

(b) *The formula for the unit root*

Theorem 4.3 *Let p be a prime with $(p, n+1) = 1$. Let*

$$f(x) = \frac{F(x)}{F(x^p)}$$

as a formal power series in $\mathbb{Z}_p[[x]]$. Then the following assertions hold true.

- (1) *$f(x)$ is in fact an element in the p -adic completion of $\mathbb{Z}_p[x, (x(1-x)H(x))^{-1}]$.*
- (2) *Let $q = p^r$ and $t \in \mathbb{F}_q$, $t \neq 0, t^{n+1} \neq 1$. Put $\lambda = t^{-(n+1)}$. Let $\hat{\lambda}$ be the Teichmüller lifting of λ . If $H(\lambda) \neq 0$, then*

$$\pi_\lambda = f(\hat{\lambda})f(\hat{\lambda}^p) \cdots f(\hat{\lambda}^{p^{r-1}})$$

is the unique unit root of V_t .

Proof. (1) Let

$$R = W(\overline{\mathbb{F}}_p)[t, (t(1-t^{n+1})H(\lambda))^{-1}].$$

Let $S = \text{Spec } R$, $S_0 = \text{Spec } R/pR$, and $S^\infty = \text{Spec } \varprojlim R/p^n R$, where the projective limit runs over all $n > 0$. Choose the lifted Frobenius σ on S^∞ by taking $\sigma(t) = t^p$. The pointwise defined $\mathcal{M}_{cris,t}$ in §2(b) forms a Hodge F -crystal \mathcal{M}_{cris} of rank n on S^∞ . As a sheaf of modules, \mathcal{M}_{cris} is isomorphic to the subsheaf \mathcal{M}_{dR} of the relative de Rham cohomology $R^{n-1}v_*^\infty \Omega_{\mathcal{V}^\infty/S^\infty}^\bullet$ of the Dwork family \mathcal{V}^∞ over S^∞ (see [6], §7). Let ϕ be the absolute Frobenius on \mathcal{M}_{cris} with respect to σ .

Over each geometric point t of S_0 , the Newton polygon of the crystal $\mathcal{M}_{cris,t}$ begins with a segment of slope zero of length 1 (Theorem 4.2). On the other hand, the absolute Frobenius ϕ on the Hodge filtration $\text{Fil}^1 \subset \mathcal{M}_{cris}$ satisfies

$$\phi(\text{Fil}^1) \subset p \cdot \mathcal{M}_{cris}$$

since \mathcal{M}_{cris} is from geometry (see [6], §7). Therefore the unit root sub-crystal U of \mathcal{M}_{cris} is generated over $W(\overline{\mathbb{F}}_q)$ by horizontal sections (op.cit., 4.1.2). Notice that the Picard-Fuchs equation (9) has a unique power series solution $F(\lambda)$ in $W(\overline{\mathbb{F}}_p)[[\lambda]]$ with constant term 1. Therefore by Corollary 3.5, the crystal U is generated by u defined in Theorem 3.4. Thus the series $f(\lambda)$ is an element in $\varprojlim R/p^n R$ (op.cit., 4.1.9). Since $f(\lambda)$ depends only on λ and its coefficients are p -adic integers, the assertion follows.

(2) Since U is generated over $W(\overline{\mathbb{F}}_q)$ by u , there exists an element $c \in W(\overline{\mathbb{F}}_p)$ such that cu is fixed by the Frobenius and $\varepsilon f(\hat{\lambda})$ (with $\varepsilon = c^{1-\sigma}$) represents the unit root of

the absolute Frobenius on $\mathcal{M}_{cris,\lambda}$ with respect to some bases over $W(\mathbb{F}_q)$ (cf. [6], §8). Thus we obtain

$$\pi_\lambda = \varepsilon^{1+\sigma+\dots+\sigma^{p^{r-1}}} f(\hat{\lambda})f(\hat{\lambda}^p)\dots f(\hat{\lambda}^{p^{r-1}}).$$

Let

$$\varepsilon' := \varepsilon^{1+\sigma+\dots+\sigma^{p^{r-1}}}$$

be the constant term above. We now ought to show that $\varepsilon' = 1$.

We apply the results in [1] (cf. op.cit., §6(j) for $n = 3$). In our case, the nilpotent part of the local monodromy (over characteristic 0) near $t = \infty$ with respect to some bases is given by the matrix

$$\mathcal{N} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

which is nilpotent of order n ([3], Corollary 1.7). By Lemma (6.2) in [1], there exist scalar $(n \times n)$ matrices A, Θ such that

$$A\mathcal{N} = q\mathcal{N}A \tag{18}$$

$$\Theta\mathcal{N}^t + \mathcal{N}\Theta = 0 \tag{19}$$

$$q^{n-1}\Theta = A\Theta A^t \tag{20}$$

and such that ε' is the (n, n) -th component of A . Under the conditions (18), (19) and (20), one finds that

$$A = \begin{pmatrix} q^{n-1}\varepsilon' & * & * & * \\ 0 & q^{n-2}\varepsilon' & * & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \dots & \varepsilon' \end{pmatrix}$$

and $\varepsilon' = \pm 1$. By op.cit. Lemma (3.4)(i),

$$f(x) \equiv H(x) \pmod{p}.$$

By the second part of Theorem 4.2, we must then have $\varepsilon' = 1$. □

Proposition 4.4 *Let*

$$F_i(x) = \frac{d^i}{dx^i} F(x)$$

be the i -th derivative of $F(x)$. Then the series

$$f_i(x) = \frac{F_i(x)}{F(x)}$$

are in fact elements in the p -adic completion of $\mathbb{Z}_p[x, (x(1-x)H(x))^{-1}]$ for all $i \geq 0$.

Proof. We keep the notations as in the proof of Theorem 4.3. Write

$$u = \sum_{i=0}^{n-1} C_i \eta^{(n-1-i)}.$$

Since the unit crystal U is generated by u , it follows that C_i/C_0 are elements in $\varprojlim R/p^n R$ for all $0 \leq i \leq (n-1)$ ([6], 4.1.9). By the explicit description of u (see the Remark after Theorem 3.4), we see inductively that $f_i(\lambda)$ is in $\varprojlim R/p^n R$. Since $f_i(\lambda)$ depends only on λ and has p -adic integral coefficients, the assertion follows for $0 \leq i \leq n-1$.

Since $F(x)$ is a solution to (9), the higher derivatives $F_i(x)$ can be written as a $\mathbb{Z}_p[x, (x(1-x))^{-1}]$ -combination of $\{F_i(x)\}_{0 \leq i \leq n-1}$. Thus the assertion also holds for all $i \geq 0$. \square

5 Method of Stienstra and Beukers

In this section, we study the unit root of V_t from the point of view of formal groups. Recall that A_m is the coefficient of $(X_1 \cdots X_{n+1})^m$ in $\mathcal{P}_t(X)^m$.

(a) *The formal group laws*

Lemma 5.1 *Let $\lambda = t^{-(n+1)}$. As polynomials in t , we have*

$$A_m = (-(n+1)t)^m {}_{n+1}F_n \left(\begin{matrix} \frac{-m}{n+1}, \frac{-m+1}{n+1}, \dots, \frac{-m+n}{n+1} \\ 1, 1, \dots, 1 \end{matrix}; \lambda \right).$$

Proof. We have

$$\begin{aligned} A_m &= \sum_{r \geq 0} \left(\prod_{i=0}^n \binom{m-ri}{r} \right) (-(n+1)t)^{m-r(n+1)} \\ &= (-(n+1)t)^m \sum_{r \geq 0} \binom{m}{(n+1)r} \frac{((n+1)r)!}{(-(n+1))^{(n+1)r} (r!)^{n+1}} \left(\frac{1}{t^{n+1}} \right)^r \\ &= (-(n+1)t)^m {}_{n+1}F_n \left(\begin{matrix} \frac{-m}{n+1}, \frac{-m+1}{n+1}, \dots, \frac{-m+n}{n+1} \\ 1, 1, \dots, 1 \end{matrix}; \lambda \right). \end{aligned}$$

\square

Proposition 5.2 *Consider the family defined by equation (5) over a noetherian ring \mathcal{A} , which is flat over \mathbb{Z} . Let $t \in \mathcal{A}$ and $\lambda = t^{-(n+1)}$. The formal group $H_{et}^{n-1}(V_t, \hat{\mathbb{G}}_m)$ associated to V_t can be realized as the formal group law G_t over \mathcal{A} with logarithm $l(\tau) \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{A}[[\tau]]$ given by*

$$l(\tau) = \sum_{m=0}^{\infty} (-(n+1)t)^m {}_{n+1}F_n \left(\begin{matrix} \frac{-m}{n+1}, \frac{-m+1}{n+1}, \dots, \frac{-m+n}{n+1} \\ 1, 1, \dots, 1 \end{matrix}; \lambda \right) \frac{\tau^{m+1}}{m+1}.$$

Proof. The formal group law with logarithm

$$\sum_{m=0}^{\infty} A_m \frac{\tau^{m+1}}{m+1}$$

realizes the formal group associated to V_t ([12], Theorem 1). Thus the statement follows by Lemma 5.1. \square

Remark. By varying t , the formal groups G_t above can be put together as a flat family of formal groups over the base space $t \in \mathbb{P}^1$. When $t = \infty$, the formal group G_∞ associated to

$$V_\infty : -(n+1)X_1X_2 \cdots X_{n+1}$$

is given by the logarithm

$$l(\tau) = \sum_{m=0}^{\infty} (-(n+1))^m \frac{\tau^{m+1}}{m+1}.$$

Thus

$$\begin{aligned} G_\infty(x, y) &= l^{-1}(l(x) + l(y)) \\ &= x + y + (n+1)xy. \end{aligned}$$

Via the transformation $x \mapsto (n+1)x$, the group G_∞ is isomorphic over \mathcal{A} to the standard multiplicative formal group

$$\hat{\mathbb{G}}_m(x, y) = x + y + xy.$$

If the residue field of \mathcal{A} at a closed point is a finite field \mathbb{F}_q , then the Frobenius endomorphism on the reduction of G_∞ at that point acts as multiplication by q .

(b) *Proofs of main results in §4 via formal groups*

In what follow, let \mathcal{R} be the p -adic completion of the ring $\mathbb{Z}_p[x, (x(1-x)H(x))^{-1}]$. Let σ be the endomorphism of \mathcal{R} extending the Frobenius on the constants and with $\sigma(x) = x^p$. For $a \in \mathcal{R}$, we write $a^\sigma = \sigma(a)$.

To facilitate the discussion, we write

$$F_{m,s}(x) = F^{<mp^s}(x)$$

for the truncated hypergeometric series up to degree $(mp^s - 1)$. Recall that $H(x) = F_{1,1}(x)$ is the Hasse invariant. Let

$$G_{\mu,s}(x) = {}_{n+1}F_n \left(\begin{matrix} \frac{-\mu p^s + 1}{n+1}, \frac{-\mu p^s + 2}{n+1}, \dots, \frac{-\mu p^s + n + 1}{n+1} \\ 1, 1, \dots, 1 \end{matrix} ; x \right).$$

Note that it is a polynomial of degree $\left[\frac{\mu p^s - 1}{n+1}\right]$. Here $[z]$ denotes the least integer function. Also write (with $\lambda = t^{-(n+1)}$)

$$G'_{\mu,s}(t) = (-(n+1)t)^{\mu p^s - 1} \cdot G_{\mu,s}(\lambda)$$

for the coefficient of $\frac{\tau^{\mu p^s}}{\mu p^s}$ in the logarithm $l(\tau)$ of G_t in Proposition 5.2. We regard $G'_{\mu,s}(t)$ as a polynomial in t .

Let \mathbb{C}_p be the p -adic completion of an algebraic closure of \mathbb{Q}_p . Take a non-archimedean norm $|\cdot|$ on \mathbb{C}_p .

Lemma 5.3 *Regarding H and $G_{\mu,s}$ as elements in \mathcal{R} , we have*

- (i) *If $1 \leq \mu \leq n+1$, then $G_{\mu,1} \equiv H \pmod{p}$.*
- (ii) *$G_{\mu,s+1} \equiv G_{\mu,0}^{\sigma^{s+1}} \cdot H^{1+\sigma+\dots+\sigma^s} \pmod{p}$.*
- (iii) *There exists an element $g \in \mathcal{R}$ such that $G_{\mu,s+1} \equiv g \cdot G_{\mu,s}^\sigma \pmod{p^{s+1}}$ for all $\mu, s \geq 0$.*
- (iv) *For any $\lambda \in \mathbb{C}_p$, if $|H(\lambda)| = 1$, then $|G_{\mu,s}(\lambda)| = 1$.*

Proof. (i) is obvious by the observation that $G_{\mu,1}(\lambda)$ has degree $\left[\frac{\mu p - 1}{n+1}\right] < p$ if $\mu \leq n+1$.

(ii) and (iii) are direct consequences of results in [13]. Let \mathcal{S} be the p -adic completion of the ring $\mathbb{Z}_p[t, (t(1-t^{n+1})H(\lambda))^{-1}]$. Let σ be the Frobenius on \mathcal{S} with $t^\sigma = t^p$. Notice that by definition, $t^{p-1}H(\lambda)$ is invertible in \mathcal{S} . Thus the reduction of the formal group law defined in Proposition 5.2 to any point of \mathcal{S} of characteristic p is of multiplicative type (see [13], Theorem (A.8)(v)). This implies (loc.cit.) that there exists an element $g' \in \mathcal{S}$ such that

$$G'_{\mu,s+1} \equiv g' \cdot (G'_{\mu,s})^\sigma \pmod{p^{s+1}}.$$

Thus

$$(-(n+1))^{\mu p^s(p-1)} t^{p-1} G_{\mu,s+1} \equiv g' \cdot G_{\mu,s}^\sigma \pmod{p^{s+1}}.$$

Since $(-(n+1))^{\mu p^s(p-1)} \equiv 1 \pmod{p^{s+1}}$, we have

$$G_{\mu,s+1} \equiv \frac{g'}{t^{p-1}} G_{\mu,s}^\sigma \pmod{p^{s+1}}.$$

Let $g = g'/t^{p-1}$. Then g depends only on λ and hence it is obvious that indeed $g(x) \in \mathcal{R}$.

Since $g \equiv G_{1,1} \equiv H \pmod{p}$, inductively we get (ii).

For (iv), assume that $|H(\lambda)| = 1$. If $\mu < n+1$, then $G_{\mu,0} = 1$. Thus $|G_{\mu,s}(\lambda)| = 1$ by (ii). In general, we can choose some ε such that $G_{\mu,s+\varepsilon} = G_{\mu',s'}$ with $0 \leq \mu' < n+1$. Since

$$G_{\mu,s+\varepsilon} \equiv G_{\mu,s}^{\sigma^\varepsilon} \cdot H^{1+\sigma+\dots+\sigma^{\varepsilon-1}} \pmod{p},$$

the equality $|G_{\mu',s'}(\lambda)| = 1$ implies $|G_{\mu,s}(\lambda)| = 1$. □

The second proof of Theorem 4.2.

Take a lifting $\hat{t} \in W(\mathbb{F}_q)$ of t and let $\hat{\lambda} = \hat{t}^{-(n+1)}$. Then $G_{\hat{t}}$ constructed in Proposition 5.2 is a formal group over $W(\mathbb{F}_q)$ whose reduction to \mathbb{F}_q is the formal group G_t associated to V_t . The group G_t is of height one if and only if the coefficient of τ^p/p in the logarithm $l(\tau)$ of $G_{\hat{t}}$ is invertible in $W(\mathbb{F}_q)$. By Lemma 5.3(i),

$${}_{n+1}F_n \left(\begin{matrix} \frac{-m}{n+1}, \frac{-m+1}{n+1}, \dots, \frac{-m+n}{n+1} \\ 1, 1, \dots, 1 \end{matrix} ; \hat{\lambda} \right) = G_{1,1}(\hat{\lambda}) \equiv H(\lambda) \pmod{p}$$

and hence the first assertion follows.

The second assertion follows from Lemma 5.3(i). Notice that the remark after Theorem 4.2 also follows easily by the same argument. \square

The second proof of Theorem 4.3.

(1) Observe that as $s \rightarrow \infty$, the elements $G_{\mu,s}(x)$ converge to $F(x)$ p -adically. Therefore we see that the element $g \in \mathcal{R}$ in Lemma 5.3(iii) must be

$$\frac{F(x)}{F(x)^\sigma} = \frac{F(x)}{F(x^p)} = f(x). \quad (21)$$

(2) This is an exercise on formal group theory and all we need is already in [13]. Let \hat{t} be the Teichmüller lifting of t . Let $a = \hat{t}^{p-1} f(\hat{\lambda})$ and

$$a_s = a^{1+\sigma+\dots+\sigma^{s-1}}.$$

Consider the formal group law G' over $W(\mathbb{F}_q)$ with logarithm

$$l'(\tau) = \tau + \sum_{s \geq 1} a_s \frac{\tau^{p^s}}{p^s}.$$

Notice that the coefficients of $l'(\tau)$ satisfy $a_{s+1}/a_s^\sigma = a$ for any $s \geq 0$. Thus the formal groups G and G' are strictly isomorphic to each other over $W(\mathbb{F}_q)$ ([13], Theorems (A.8) and (A.9)).

On the other hand, the group G' is isomorphic to a formal group whose Cartier module (which is of rank one over $W(\mathbb{F}_q)$) has a basis ω with the Frobenius acting as $\omega \mapsto a\omega$ (op.cit. (A.13)). Thus the p -adic unit

$$f(\hat{\lambda})f(\hat{\lambda}^p) \dots f(\hat{\lambda}^{p^{r-1}}) = aa^\sigma \dots a^{\sigma^{r-1}}$$

equals the Frobenius endomorphism of the Cartier module of G . Hence it is the unique eigenvalue of the geometric Frobenius endomorphism on the middle cohomology of V_t . \square

Remark. The formula for the unit root also makes sense when $\lambda = 0$ if one consider the variation of the Frobenius endomorphism on the Cartier module of the formal groups G_t

associated to V_t . See the remark after Proposition 5.2.

(c) *Dwork's congruences*

Here we remark some congruent relations.

Write $F(x) = \sum B(i)x^i$. Combine Lemma 5.3 (i) and (ii), we find

$$F(x) \equiv F^{<p}(x)F(x^p) \pmod{p}.$$

Comparing the coefficients on both sides, this implies for $0 \leq c < p$, we have

$$B(c+p) \equiv B(c)B(1) \pmod{p},$$

which is a tiny special case of [1], §1, Corollary 2.

In [1], Lemma (3.4), one finds the congruences

$$F_{m,s+1}(x) \cdot F(x^p) \equiv F_{m,s}(x^p) \cdot F(x) \pmod{p^{s+1}}$$

for any integers $m, s \geq 0$. Thus by Lemma 5.3 and (21), we have that

$$F_{m,s+1}(x) \cdot G_{\mu,s}(x^p) \equiv F_{m,s}(x^p) \cdot G_{\mu,s+1}(x) \pmod{p^{s+1}} \quad (22)$$

for any non-negative integers m, μ, s as polynomials in $\mathbb{Z}[\frac{1}{n+1}][x]$. Consider the formal group laws J_λ over $\mathbb{Z}_p[\lambda]$ and J_t over $\mathbb{Z}_p[t, t^{-1}]$ with logarithms

$$j_\lambda = \sum F_{m,s}(\lambda) \frac{\tau^{mp^s}}{mp^s} \quad \text{and} \quad j_t = \sum t^{mp^s-1} F_{m,s}(\lambda) \frac{\tau^{mp^s}}{mp^s},$$

respectively. Then the congruences (22) imply that J_t is strictly isomorphic to G_t over $\mathbb{Z}_p[t, t^{-1}]$. It would be interesting to see if one can find some geometry behind these formal groups and its relation to the Dwork family V_t .

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