Hesse cubics and GIT-stability

Iku Nakamura*

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Abstract: The moduli space of nonsingular projective curves of genus g is compactified by the moduli of Deligne-Mumford stable curves of genus g. We compactify in a similar way the moduli space of abelian varieties as the moduli space of some mildly degenerating limits of abelian varieties.

A typical case is the moduli space of Hesse cubics

$$x_0^3 + x_1^3 + x_2^3 - \mu x_0 x_1 x_2 = 0, \quad (\mu \in \mathbf{P}^1).$$

For $\mu^3 = 1$ or $\mu = \infty$, it is a 3-gon. Otherwise it is a nonsingular elliptic curve, therefore, it is an abelian variety of dimension one. These curves have a simple property in common : they are invariant under the transformations

$$\sigma : x_k \mapsto \zeta_3^k x_k, \quad \tau : x_0 \to x_1 \to x_2 \to x_0,$$

where $\zeta_3 = \frac{-1+\sqrt{-3}}{2}$, $\zeta_3^3 = 1$. This is the key property for all the rest.

Any Hesse cubic is GIT-stable by the above property, and any GIT stable planar cubic is one of Hesse cubics. Similarly in arbitrary dimension, the moduli space of abelian varieties is compactified by adding only GIT-stable limits of abelian varieties. Our moduli space is a projective "fine" moduli space of (possibly degenerate) abelian schemes

with non-classical (non-commutative) level structure

over $\mathbb{Z}[\zeta_N, 1/N]$ for some $N \geq 3$.

*Hokkaido University