Seminar on "Stability Conditions and Stokes Factors"

October 11, 2012

The aim of this seminar is to study the wall-crossing phenomena via the language of Stokes factors appeared in the theory of ordinary differential equations (with irregular singularities). We are going to go through the materials in the two papers [3, 2]. Along the journey, we will also learn (very explicitly) the Stokes phenomena, the isomonodromic deformation and the multilogarithms in the simplest cases, as well as the Ringel-Hall algebra appeared in the structure of abelian category with a stability condition. In this way we also see a very close connection between the quite new topic (stability, wall-crossing) and the classical objects (Stokes, multilogs) Hopefully this new aspect can help understand the wall-crossing phenomena and the space of stability conditions.

For more detailed description of the project, see [2, Introduction]. The following is the tentative plan of the talks.

(a) Stability side [2 ∼ 3 talks]

The aim is to explain the concepts and prove the formulas listed in the table (except the last row) of the next section. The key is to understand the definition of the Ringel-Hall algebra (to study the (semi-)stable objects), the basic characteristic elements in it, and the relations among them (e.g. obtained from Harder-Narasimhan filtration).

One can either follow [4, §§2-5] and work on the space of representations *over a finite field* of a certain quiver (i.e. concentrate on the "Reineke" column of the table). However in the application, we will work on the representation space *over the field* $\mathbb{C}$ of a certain $\mathbb{C}$-algebra. Thus it is better to axiomatize what are needed to derive the main results in [4] so that one can use them in another setting with confidence later in this seminar. If one is interested, it is also good to finish the whole paper [4].

Or one can simply go through [2, §4 - §5] and prove the statements there by modifying the proofs of the corresponding results in [4] (i.e. concentrate on the "B-TL" column). One may skip the materials in §4.4 - §4.8 here since they will only appeared in the construction of the connection on $\mathbb{P}^1$ later (serving as the structure group of the principal bundle). We are not going to go through the structures of the moduli stack of the representations underlying the definition of the algebra structure of $\mathcal{H}(\mathcal{A})$ in this seminar. They will only be recalled and assumed.

(b) ODE side [4 talks]

The goal is to study the paper [3]. Here we learn the Stokes structure attached to an ODE on $\mathbb{P}^1$, the Fourier-Laplace transform, and the isomonodromic deformation under the simplest irregular singular situation. The multilogaritms then appear naturally in the explicit construction of the Riemann-Hilbert correspondence in this setting. We shall (1) derive the explicit formulas for the Stokes and the inverse Stokes maps (note that the answer involves Lie series!), and (2) study the isomonodromic equation. There are four talks consisting of

(i) Regular-irregular correspondence via Fourier-Laplace (§§6,7,8): Thm.2.5 for GL(V), parallel transport via iterated integral. The multilogarithms appear here.

(ii) Stokes factors via Fourier-Laplace (§9, part of §10): Thm.4.5 for GL(V).

(iii) Stokes via Tannaka and the inverse Stokes (§10,11): Thm.2.5, 4.5, 4.7, 4.8. The latter two theorems involve the universal Lie series. Here we generalize the formula of Stokes factors to the case of connections on a trivial principal G-bundle. The other main purpose is to derive the formula for the inverse of the Stokes map.

(iv) Isomonodromy: §3, Thm.4.9. The aim is to derive the isomonodromic differential equations and to understand their geometric meaning. The statements can be found in [3, §3], where the authors refer to [1] for the proofs. It would be great if we can work out the details in some extend in the simple setting that we need.

(*c*) *Tie-up [1 $\sim$ 2 talks]*

Finally we explain the notations of the space of stability conditions and the wall-crossing. Then we formulate and prove the main results restated here.

Theorem 1 *Let Z be the variable parametrizing stability conditions on the abelian category $\mathcal{A}$. Consider the connection on $\mathbb{P}^1$*

$$d - \left(\frac{Z}{t^2} + \frac{f}{t}\right) dt$$

on the trivial principal $\widehat{B}(\mathcal{A})$-bundle attached to the Stokes factors SS_ℓ via the inverse Stokes map. Then, as Z varies, the connection is an isomonodromic deformation.

Proof. Follows from the equation

$$\overset{\curvearrowright}{\prod_{\ell}} \mathrm{SS}_\ell = 1_{\mathcal{A}}.$$

□

Corollary 2 *Joyce's system of differential equations*

Dictionary

Besides the explicit formula of the Riemann-Hilbert correspondence, the main ingredient of the proof is the formulations in [4] concerning the relations of natural elements in the Ringel-Hall algebra, which encodes the information of the (semi-)stable objects and the

Harder-Narasimhan filtration of a certain category. Since the papers [4] and [2] work on differential settings and use different notations, we provide the table of nations used there for comparison.

	Reineke	B-TL
Inputs	Q, a certain quiver	$\mathcal{A}$, a certain abelian category
Parameters	dimension vector d	dimension d and elements γ in the K-group $K(\mathcal{A})$
Algebras[a]	$\mathcal{H}_d(Q) = \mathbb{C}^{G_d}[R_d]$ $\mathcal{H}(Q) = \bigoplus_d \mathcal{H}_d(Q)$ $\mathcal{C}(Q) = \langle \chi_i \mid i \in I \rangle$	$\mathcal{H}_d(\mathcal{A}) = \mathbb{C}^{GL_d(\mathbb{C})}[\mathrm{Rep}_d]$ $\mathcal{H}(\mathcal{A}) = \bigoplus_d \mathcal{H}_d(\mathcal{A})$ $= \bigoplus_{\gamma \in K(\mathcal{A})} \mathcal{H}_\gamma(\mathcal{A})$ $\mathcal{C}(\mathcal{A}) = \langle \kappa_\gamma \mid \gamma \in K(\mathcal{A}) \rangle$
Elements[b]	$\begin{matrix} \chi_d \\ \chi_d^{ss} \end{matrix} \in \mathcal{H}_d(Q)$	$\begin{matrix} \kappa_\gamma \\ \delta_\gamma \end{matrix} \in \mathcal{H}_\gamma(\mathcal{A})$
Extra[c]		$\epsilon_\alpha = \sum \frac{(-1)^{n-1}}{n} \delta_{\gamma_1} * \cdots * \delta_{\gamma_n}$ $\big(\in \mathfrak{n}(\mathcal{A}) \subset \mathcal{C}(\mathcal{A}) \big)$
Relations[d]	$\chi_d = \chi_d^{ss} + \sum v^{-\langle d^* \rangle} \chi_{d^1}^{ss} * \cdots * \chi_{d^s}^{ss}$ $\chi_d^{ss} = \sum (-1)^{s-1} v^{-\langle d^* \rangle} \chi_{d^1} * \cdots * \chi_{d^s}$	$\kappa_\gamma = \sum \delta_{\gamma_1} * \cdots * \delta_{\gamma_n}$ $\delta_\gamma = \sum (-1)^{n-1} \kappa_{\gamma_1} * \cdots * \kappa_{\gamma_n}$
Series[e]	$X(T) := \sum \chi_d * T^d$ $X_\mu^{ss}(T) := \sum \chi_d^{ss} * T^d$	$1_{\mathcal{A}} := \sum \kappa_\gamma$ $\mathrm{SS}_\ell := 1 + \sum \delta_\gamma$ $\Big(= \exp \{ \sum \epsilon_\alpha \} \Big)$
Formulas[f]	$X(T) = \overleftarrow{\prod_{\mu \in \mathbb{Q}}} X_\mu^{ss}(T)$	$1_{\mathcal{A}} = \overset{\curvearrowright}{\prod_{\ell}} \mathrm{SS}_\ell$
Extra[g]		Ringel-Hall Lie algebra $\mathfrak{n}(\mathcal{A}), \mathfrak{b}(\mathcal{A})$ Completed one $\widehat{\mathfrak{n}}(\mathcal{A}), \widehat{\mathfrak{b}}(\mathcal{A})$ Ringel-Hall group $\widehat{N}(\mathcal{A}), \widehat{B}(\mathcal{A})$ $\exp : \widehat{\mathfrak{n}}(\mathcal{A}) \longleftrightarrow \widehat{N}(\mathcal{A}) : \log$

[a] [4, Def.4.1, 4.3], [2, §4.2]
[b] [4, Def.4.3, 4.6], [2, §4.3, §5.3]
[c] [2, §5.8]
[d] [4, Prop.4.8, Thm.5.1], [2, Thm.5.4]
[e] [4, Del.4.11], [2, §5.5, §5.6, §5.8]
[f] [4, Prop.4.12], [2, §5.7]
[g] [2, §§4.4-4.8]

References

[1] P. Boalch, Symplectic manifolds and isomonodromic deformations. *Adv. Math.* 163 (2001), no. 2, 137-205.

[2] T. Bridgeland and V. Toledano Laredo, Stability conditions and Stokes factors. *Invent. Math.* 187 (2012), no. 1, 61-98.

[3] T. Bridgeland and V. Toledano Laredo, Stokes factors and multilogarithms. To appear in *Crelle's*.

[4] M. Reineke, The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli. *Invent. Math.* 152 (2003), no. 2, 349-368.