Theory of Computation

Fall 2017, Midterm Exam. Solutions (Nov. 7, 2017)

1. (20 pts) Let $\Sigma = \{0, 1\}$, answer the following questions (True or False) and prove your answer:
   (a) the set of nonpalindromes (i.e., $\Sigma^* - \{w \mid w = w^R, w \in \Sigma^*\}$) is nonregular;
   Solution: True. $L \cap 1^*01^* = \{1^n 0 1^n \mid n \geq 0\}$ is nonregular.
   (b) the set of odd-length strings with middle symbol 0 is regular;
   Solution: False. $L \cap 1^*01^* = \{1^n 0 1^n \mid n \geq 0\}$ is nonregular.
   (c) the set of strings that contain a substring of the form wuw where $u \in \Sigma^*, w \in \Sigma^+$ is nonregular;
   Solution: False. $L = \bigcup_{a \in \Sigma} \Sigma^* a \Sigma^* a \Sigma^*$
   (d) the set of strings with the property that in every prefix, the number of 0s and the number of 1s differ by at most 2 is regular;
   Solution: True. Use the state to keep the number of differences of 0s and 1s, which involves a finite number of cases.
   (e) if L is nonregular and both of L' and $L \cap L'$ are regular, then $L \cup L'$ is nonregular.
   Solution: True. $L = (L \cup L') - (L' - (L \cap L'))$. If $L \cup L'$ is regular, so is L.

2. (16 pts) Let $A = \{xx \mid x \in \{a, b\}^*\}$, and $h : \{a, b\}^* \to \{a, b\}^*$ be a homomorphism with $h(a) = h(b) = a$.
   (a) What is $h(A)$?
   Solution: $\{a^{2n} \mid n \geq 0\}$
   (b) What is $h^{-1}(A)$?
   Solution: $\{x \mid x \text{ is of even length}\}$
   (c) What is $h^{-1}(h(A))$?
   Solution: $\{x \mid x \text{ is of even length}\}$
   (d) What is $h(h^{-1}(A))$?
   Solution: $\{a^{2n} \mid n \geq 0\}$

3. (9 pts) Given $\Sigma = \{a, b\}$, we define $Two(x)$ to be an operation doubling each symbol in $x \in \Sigma^*$. For instance, $Two(abab) = aabbaabb$, $Two(aab) = aaaabb$.
   (a) Define $Two(x)$ recursively.
   Solution: $Two(\epsilon) = \epsilon$; $Two(aw) = aa\,Two(w)$, $\forall a \in \Sigma, w \in \Sigma^*$.
   (b) Given a language L, define $Two(L) = \{x \mid Two(x), x \in L\}$. Prove that if L is regular, so is $Two(L)$.
   Solution: Define a homomorphism $h(a) = aa, h(b) = bb$.

4. (5 pts) Consider the following operations:
   $prefix(L) = \{u \mid uv \in L, \exists v \in \Sigma^*\}$; $suffix(L) = \{v \mid uv \in L, \exists u \in \Sigma^*\}$; $reverse(L) = \{x \mid x^R \in L\}$.
   Use the closure of regular languages under the reverse and prefix operations to prove that $suffix(L)$ is regular whenever L is regular.
   Solution: $suffix(L) = reverse(prefix(reverse(L)))$

5. (5 pts) Use the Myhill-Nerode theorem to show that for any positive integer m, no DFA with less than m states recognizes $A_m = \{1^k | m \text{ divides } k\}$ $(\subseteq \{1\}^*)$.
   Solution: A DFA with m states which simply stores the number of 1s seen so far, modulo m recognizes this language. Also, for any two strings 1^{k_1} and 1^{k_2} such that $k_1 \neq k_2 \mod m$, the string $1^{m-(k_1 \mod m)}$ distinguishes the two. Hence, any two strings in which the number of 1s is different modulo m must be in different equivalence classes, showing that no DFA with less than m states can recognize this language.

6. (10 pts) Let L be an infinite regular language. Prove that L can be partitioned into two disjoint infinite regular languages, i.e., $L = L_1 \cup L_2$, $L_1 \cap L_2 = \emptyset$, and L_1, L_2 are infinite regular languages. (Hint: Use the pumping lemma.)
   Solution: By the pumping lemma, there is $p > 0$ such that every string $w \in L$ of length at least p can be written as $w = xyz$, where y is nonempty and $xy^iz \in L$ for all $i \geq 0$, for all $i \geq 0$.

   So, fix an arbitrary string $w \in L$ of length at least p (it exists because L is infinite). Let $w = xyz$ be a decomposition guaranteed by the pumping lemma. Partition $L = A \cup (L - A)$; where $A = \{xy^iz \mid i = 0, 2, 4, 8, ...\}$.

   - DISJOINTNESS: Trivial
   - INFINITENESS: Trivial
   - REGULARITY: A is regular because it is given by a regular expression, $x(yy)^*z$, which makes $L - A$ regular as well by the closure properties.

7. (10 pts) Consider the following grammar G, where S, A are nonterminals, and a, b are terminals:
$S \rightarrow aSA \mid \epsilon \quad ; \quad A \rightarrow bA \mid \epsilon$
Answer the following questions:

   (a) Is $L(G)$ regular? Why?
   Solution: Yes. $L = a^+b^* \cup \{\epsilon\}$.

   (b) Is G ambiguous? Explain your answer.
   Solution: Yes the grammar is ambiguous.
   $S \Rightarrow aSA \Rightarrow aaSAA \Rightarrow aaAA \Rightarrow aabAA \Rightarrow aabbAA \Rightarrow aabbA \Rightarrow aabb$, and
   $S \Rightarrow aSA \Rightarrow aaSAA \Rightarrow aaAA \Rightarrow aaA \Rightarrow aabA \Rightarrow aabbA \Rightarrow aabb$;
   their corresponding parse trees are easy to construct.

8. (10 pts) True or False? Score = max{0, Right - $\frac{1}{2}$ Wrong}. No explanations are needed.
Here are four regular expressions over the alphabet $\{a, b\}$:
$E_1 = (ab + a^*b^*b^*)^* \quad E_2 = ((ab)^*(a^*b^*b^*)^*)^* \quad E_3 = (a + b)^* \quad E_4 = a(a + b)^*$

   (1) $L(E_2) = L(E_3)$
   Solution: True

   (2) $L(E_3) = L(E_4)$
   Solution: False

   (3) $L(E_1) = L(E_4)$
   Solution: False

   (4) The minimal DFA for $L(E_1)$ has five states.
   Solution: False

   (5) The minimal DFA for $L(E_4)$ has two states.
   Solution: False Note: ϵ, a, b are in different equivalence classes of $R_{L(E_4)}$.

9. (10 pts) Use the pumping lemma to prove that the following language is not regular:
$L = \{0^m 1^n \mid m \leq 2n + 5, m, n \in N\}$.
Solution We will use the pumping lemma to prove that the language is not regular. Assume that L is regular and p is its pumping length. Take the word $w = 0^p 1^p$. Since $p \leq 2p + 5$ then $w \in L$. Also it is clear that $|w| = 2p \geq p$. From pumping lemma we have that $w = xyz$ where x, y and z are such that for all $i \geq 0$ it holds $xy^iz \in L$. Also $|y| > 0$ and $|xy| \leq p$. Since $|xy| \leq p$, both x and y consists of zeros only. Take $i = 2p + 6$ and form the word $xy^{2p+6}z$. According to the pumping lemma this word should belong to L. However, $|xy^{2p+6}| \geq (2p + 6)|y| \geq 2p + 6$. It means that the inequality of numbers of zeros and ones defined in L does not hold any more: $2p + 6 \not\leq 2p + 5$, i.e. $xy^{2p+6}z \notin L$. Contradiction. This means that original assumption was wrong and L is not regular.

10. (5 pts) We say that a DFA M for a language A is minimal if there does not exist another DFA M' for A such that M' has strictly fewer states than M. Suppose that $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for A. Using M, we construct a DFA $\overline{M}$ for the complement $\overline{A}$ as $\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$. Is $\overline{M}$ is a minimal DFA for $\overline{A}$? Why?
Solution: Yes. If otherwise, suppose $\dot{M}$ is a minimal DFA for $\overline{A}$ with fewer states, then $\overline{\dot{M}}$ is a minimum DFA for M, a contradiction.