Theory of Computation

Fall 2013, Midterm Exam.

Due: Nov. 4, 2013

1. (20 pts) True or False? Score = max{0, Right - $\frac{1}{2}$ Wrong}. No explanations are needed.

   (a) Let r_1 and r_2 be two regular expressions. Then $L((r_1 + r_2)^*) = L((r_1^* r_2^*)^*)$.
   Sol: True. Easy to show.

   (b) Let $L_4 = L_1 \cdot L_2 \cdot L_3$. If L_1 and L_2 are regular and L_3 is not regular, it is possible that L_4 is regular.
   Sol: True. Consider the case when $L_1 = \emptyset$. Then $L_4 = \emptyset$

   (c) $L_1 \subseteq L \subseteq L_2$ where L_1 and L_2 are regular, then L must be regular.
   Sol: False. $\emptyset \subseteq \{a^n b^n \mid n \geq 0\} \subseteq \{a, b\}^*$

   (d) $\{w = xyzy \mid x, y, z \in \{0,1\}^+\}$ is regular.
   Sol: True. The set is $\{x0z0 \mid x, z \in \{0,1\}^+\} \cup \{x1z1 \mid x, z \in \{0,1\}^+\}$ (i.e., pick $y = 0$ or 1)

   (e) $(\emptyset^* \cdot \emptyset)^* \cdot \emptyset^* = \emptyset$
   Sol: False. LHS=$\{\epsilon\}$

   (f) $\{xyx^R \mid x, y \in \{a,b\}^+\}$ is regular.
   Sol: True. The language is $\{aya \mid y \in \{a,b\}^+\} \cup \{byb \mid y \in \{a,b\}^+\}$ (i.e., pick $x = a$ or b).

   (g) The language generated by the grammar $G = (\{S\}, \{a\}, S, \{ S \to Saaa \mid aS \mid a \})$ is not regular.
   Sol: False. It is a context-free language over a single-letter alphabet $\{a\}$, which is always regular. Note that even though the grammar is not of the form of a regular grammar, the language could still be regular.

   (h) Let $L_1, L_2, ..., L_n, ...$ be a countably infinite number of regular languages. $\bigcup_{i \geq 1} L_i = L_1 \cup L_2 \cup \cdots \cup L_n \cup \cdots$ is always regular.
   Sol: False. Consider $\bigcup_{p\ is\ prime} \{a^p\}$.

   (i) Define $a \setminus L = \{w \mid aw \in L\}$. If $L = \{0, 001, 100\}$, then $0 \setminus L = \{\epsilon, 01\}$.
   Sol: True

   (j) Let h be a (homo)morphism from Σ^* to Δ^*. Given an arbitrary language $A \subseteq \Delta^*$, $h(h^{-1}(A)) = A$.
   Sol: False. A may contain a word w such that $h^{-1}(w) = \emptyset$.

2. (10 pts) Let $A = \{xx \mid x \in \{a,b\}^*\}$. Consider (homo)morphism $h : \{a,b\}^* \to \{0,1\}^*$ with $h(a) = 00; h(b) = \epsilon$.

   (a) What is $h(A)$? Why?
   Sol. $\{0^{4n} \mid n \geq 0\}$

   (b) What is $h^{-1}(h(A))$? Why?
   Sol. $\{w \in \{a,b\}^* \mid w \text{ contains an even number of } a's\}$

3. (10 pts) Find a Chomsky normal form for the language $\{0^m 1^m 0^n 1^n \mid m, n \geq 1\}$.
   $G = (\{S, M, O, I, O'\}, \{0,1\}, S, P)$, with $P =$
   $S \to MM$
   $M \to O'I; \quad O' \to OM; \quad O \to 0; \quad I \to 1$

4. (10 pts) Prove that $\{0^{n^3} \mid n \geq 0\}$ is not regular using Myhill-Nerode theorem.
   Sol.
   Let $p = (k+1)^3 - k^3 = 3k^2 + 3k + 1$. Clearly $0^{k^3} \cdot 0^p = 0^{(k+1)^3} \in L$. However, $0^{(k+1)^3} \cdot 0^p \notin L$, because its length is $k^3 + 6k^2 + 6k + 2$, but $(k+2)^3 = k^3 + 6k^2 + 12k + 8$, i.e., $0^{(k+1)^3} \cdot 0^p$ falls between 0^{k^3} and $0^{(k+1)^3}$. Hence $0^{k^3} \not\equiv_L 0^{(k+1)^3}$, since there exists a z such that $0^{k^3} z \in L$ but $0^{(k+1)^3} z \notin L$. In view of the above, 0^{k^3}, $0^{(k+1)^3}$, $0^{(k+2)^3}$ ... are pairwise nonequivalent. By Myhill-Nerode theorem, L is not regular.

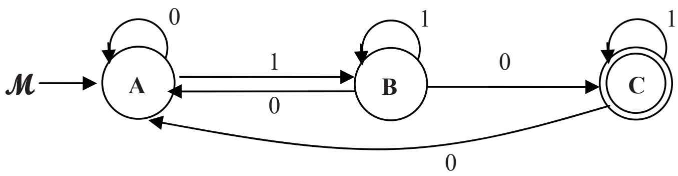
5. (10 pts) For a string $a_1 a_2 \cdots a_n$, define the operation *shift* as *shift*$(a_1 a_2 \cdots a_n) = a_2 \cdots a_n a_1$.
   For example, *shift*(*abcde*)=*bcdea*). Given a language L, *shift*(L)=$\{$*shift*$(w) \mid w \in L\}$.
   Question: Are regular languages closed under *shift*? That is, if L is regular, so is *shift*(L)? Give a formal proof.
   Sol. Yes.
   Idea: (1) store the first symbol read in the "finite-state control" which can contain any symbol of Σ , and then go to a state $q_{firststep}$ that allows the automaton to consume the first symbol while storing it in the "finite-state control". (2) consume the input string while behaving like M, (3) if the DFA is in a final state, and the next symbol read is the same as the symbol in the "finite-state control", then guess that the next symbol being read is the last symbol of the input string and go to the final state of the NFA.

Given a DFA $M = (Q, \Sigma, \delta, s, F)$ to accept L, we construct the NFA without ϵ-transitions which contains Q, new start state s^* , new final state f^* , and a new state $q_{firststep}$

$$((\Sigma \times Q) \cup \{s^*, f^*, q_{firststep}\}, \Sigma, \Delta, \{s^*\}, \{f^*\} \cup \{(a, q_{firststep}) \mid \delta(s, a) \in F\})$$

- $\Delta(s^*, a) = \{(a, q_{firststep})\}, \forall a \in \Sigma$
- $\Delta((a, q_{firststep}), b) = \{(a, \delta(\delta(s, a), b))\}, \forall a, b \in \Sigma$
- $\Delta((a, q), b) = \{(a, \delta(q, b))\}, \forall a, b \in \Sigma,\ \forall q \notin F$
- $\Delta((a, q), a) = \{(a, \delta(q, a)), f^*\}, \forall a \in \Sigma,\ \forall q \in F$
- $\Delta(f^*, a) = \emptyset, \forall a \in \Sigma$

6. (10 pts) Consider the following NFA M. Assuming that we designate A as state 1, B as state 2 and C as state 3. We associate regular expressions $R^k_{i,j}$ with M where $R^k_{i,j}$ is the set of strings formed by going from state i to state j without passing through states whose indices are higher than k. Find $R^3_{1,3}$. Show your derivation in detail.



Sol.

- $R^3_{1,3} = R^2_{1,3} \cup (R^2_{1,3}(R^2_{3,3})^* R^2_{3,3})$.
  - $R^2_{3,3} = R^1_{3,3} \cup (R^1_{3,2}(R^1_{2,2})^* R^1_{2,3})$.
    - ($R^1_{3,3} = 1,$, $R^1_{3,2} = 00^*1$, $R^1_{2,2} = (1 + 00^*1)$, $R^1_{2,3} = 0$) $\Longrightarrow R^2_{3,3} = 1 + 00^*1(1 + 00^*1)^*0$
  - $R^2_{1,3} = R^1_{1,3} \cup (R^1_{1,2}(R^1_{2,2})^* R^1_{2,3})$
    - ($R^1_{1,3} = \emptyset$, $R^1_{1,2} = 0^*1$, $R^1_{2,2} = (1 + 00^*1)$, $R^1_{2,3} = 0$) $\Longrightarrow R^2_{1,3} = 0^*1(1 + 00^*1)^*0$

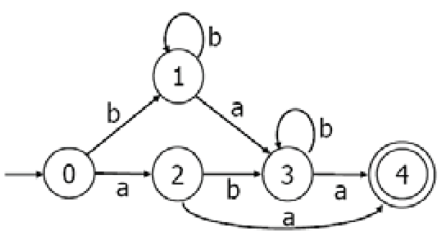
7. (20 pts) Define $Max(L) = \{w \mid w \in L,$ if $wy \in L$ then $y = \epsilon\}$ and $Min(L) = \{w \mid w \in L,$ if $(xy = w, x \in L)$ then $y = \epsilon\}$.

   (a) (10 pts) Prove that the class of regular languages is closed under Max.
   Sol. The proof is by construction. If L is regular, then it is accepted by some DFSA $M = (Q, A, \delta, s, F)$. We construct a new DFSM $M' = (Q', A, \delta', s', F')$, such that $L(M') = Max(L)$. The idea is that M' will operate exactly as M would have except that F' will include only states that are accepting states in M and from which there exists no path of at least one character to any accepting state (back to itself or to any other).

   (b) (6 pts) Consider a language $L = \{a^n b^m c^t \mid t = n \text{ or } t = m\}$. Find $Max(L)$ and $Min(L)$.
   Sol. $Max(L) = \{a^n b^m c^t \mid t = max\{m, n\}\}$;
   $Min(L) = \{a^n b^m c^t \mid t = min\{m, n\}\}$, if $m, n > 0$; $=\{\epsilon\}$, if m, n are allowed to be zero (i.e., $\epsilon \in L$).

   (c) (4 pts) If $Max(L)$ is regular, must L also be regular? Prove your answer.
   Sol. No. Consider $L = \{a^n : n \text{ is prime }\}$. L is not regular. But $Max(L) = \emptyset$, which is regular.

8. (10 pts) Let A be the following DFA. Find the minimum DFA equivalent to A. Show your derivation in detail.



Sol: