Theory of Computation

Midterm Exam, Fall 2008.

Name: ....*KEY*.................   Student I.D. ....

1. (30 pts) True or false? Mark 'O' for true; '$\times$' for false. Score=$\max\{0, Right - \frac{1}{2}Wrong\}$

   (a) ....*O*..... The set of all strings containing an even number of 0's on $\Sigma = \{0,1\}$ is regular.

   (b) ....*X*..... The set of all strings over $\Sigma = \{0,1\}$ that contain more 1's than 0's is regular.

   (c) ....*X*..... $\{a^n : n \geq 2, n$ is not a prime number$\}$ is regular.
   (Sol: $a^* - L =$ the set of all primes, which is not regular)

   (d) ....*X*..... If L is a regular language and L' is a subset of L, then L' is also regular.

   (e) ....*O*..... If $L_1 \cup L_2$ is regular and L_1 is finite, then L_2 is also regular.

   (f) ....*X*..... If $L_1, L_2, L_3, ...$ are all regular, then the language $\bigcap_{i=1}^{\infty} L_i$ is also regular.
   (Sol: let $P = \{p_1, p_2, ...,\}$ be the set of all prime, which is not regular. Let $L_i = \{p_1, ..., p_i\} \cup \{n \mid n \geq p_i\}$, each of which is regular. Then $\bigcap_{i=1}^{\infty} L_i = P$)

   (g) ....*O*..... $\{w \in \{0,1\}^* \mid w = w^R\}$ is context-free. (Here w^R denotes the reversal of string w.)
   (Sol. the language is a palindrome)

   (h) ....*O*..... $\{0^i 1^j \mid 0 \leq i = j \leq 17\}$ is regular.
   (Sol: the language is finite)

   (i) ....*O*...... If R is a regular expression that never uses the $*$ operation, then $L(R)$ is finite.

   (j) ....*X*..... If L_1 is a CFL and L_2 is not a CFL, then $L_1 \cap L_2$ is not a CFL.
   (Sol: $L_1 = \emptyset$.)

   (k) ....*X*..... For a language L and a homomorphism h, if $h(L)$ is regular, then L is regular.
   (Sol: $L = \{a^n b^n \mid n \geq 0\}$ and $h(a) = h(b) = \epsilon$, then $h(L) = \{\epsilon\}$. Note that $L \neq h^{-1}(h(L))$)

   (l) ....*X*..... $\{a^n b^{n^2} \mid n \geq 1\}$ is context-free.

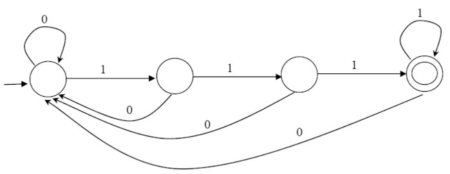
   (m) ....*O*..... $\{0^n 1^m 0^n \mid n + m \geq 2\}$ is a CFL.
   (Sol: L is the union of 111^*, 011^*0 and $\{0^n 1^* 0^n \mid n \geq 2\}$.)

   (n) ....*X*..... $\{1^k y \mid y \in \{0,1\}^*$ and y contains <u>at most</u> k 1's, for $k \geq 1\}$ is regular.
   (Sol: $L \cap 1^*01^* = \{1^k 01^m \mid k \geq m\}$)

   (o) ....*O*..... $\{1^k y \mid y \in \{0,1\}^*$ and y contains <u>at least</u> k 1's, for $k \geq 1\}$ is regular.
   (Sol: $L = 1 \cdot (\{0,1\}^* 1 \{0,1\}^*)$)

2. (10 pts) Consider the language $L = (0 \cup 1)^* 111$, i.e., the set of all strings (over alphabet $\{0,1\}$) that end in 111.

   (a) (5 pts) Draw a 4-state DFA M to accept L.

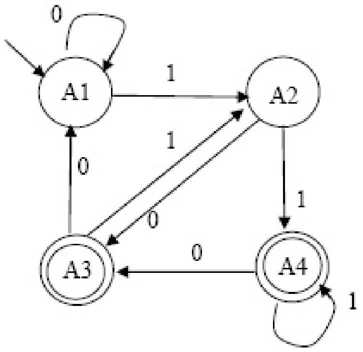


   (b) (5 pts) Prove formally that L cannot be accepted by any 3-state DFA. (Hint: Proof by contradiction.)
   (Solution) Suppose L were accepted by a 3-state DFA. Then from the initial state, two of the strings 1^i and 1^j (with $0 \leq i < j \leq 3$) in the set $\{1^k \mid k = 0,1,2,3\}$ will lead to the same state. In this case, both $1^i 1^{3-j}$ and $1^j 1^{3-j}$ should be in L (or both not in L) – a contradiction.

3. (10 pts) Given a language $L \subseteq \Sigma^*$, two words $x, y \in \Sigma^*$ are L-equivalent, written as $x \approx_L y$, iff for all words $z \in \Sigma^*$, we have $xz \in L$ iff $yz \in L$.

(a) (5 pts) Suppose $\Sigma = \{0, 1\}$, and consider language $L = \Sigma^*1\Sigma$. What are the equivalence classes induced by $\approx_L$? (For each of the equivalence class, write a regular expression to specify the set of words associated with the equivalence class.)
(Solution) $A_1 = \Sigma^*00 \cup 0 \cup \epsilon$; $A_2 = \Sigma^*01 \cup 1$; $A_3 = \Sigma^*10$; $A_4 = \Sigma^*11$

(b) (5 pts) Draw a minimal DFA to accept the above langauge L.



4. (20 pts) Answer the following questions:

(a) (5 pts) Give a context-free grammar for the language $L_1 = \{a^i b^{2i+1} \mid i \geq 0\}$, where $\Sigma = \{a, b\}$.
(Solution)
$S \rightarrow Xb$; $X \rightarrow aXbb \mid \epsilon$

(b) (5 pts) Give a context-free grammar for the language $L_2 = \{a^m b^n c^p d^q \mid m + n = p + q\}$, where $\Sigma = \{a, b, c, d\}$. (Hint: Design four nonterminals T, X, Y, Z that generate $a^i T d^i$, $a^j X c^j$, $b^r Y d^r$, $b^t Z c^t$, respectively.)
(Solution)
$S \rightarrow T \mid X \mid Y \mid Z$
$T \rightarrow aTd \mid X \mid Y \mid \epsilon$
$X \rightarrow aXc \mid Z \mid \epsilon$
$Y \rightarrow bYd \mid Z \mid \epsilon$
$Z \rightarrow bZc \mid \epsilon$

(c) (5 pts) Given two languages L_1 and L_2, let $L_1/L_2 = \{w \mid \exists x, (x \in L_2) \wedge (wx \in L_1)\}$. Suppose $L_1 = \{a^n b^n c^n \mid n \geq 0\}$ and $L_2 = \{b^i c^j \mid i, j \geq 0\}$. What is L_1/L_2? Why?
(Solution) $\{a^p b^q c^r \mid (p = q \geq r) \vee (p \geq q \wedge r = 0)\}$

(d) (5 pts) If D is any language over Σ, define $D_{\frac{1}{3}-\frac{1}{3}} = \{xz \mid \exists y \in \Sigma^*, |x| = |y| = |z|, xyz \in D\}$. Prove that for a regular language D, $D_{\frac{1}{3}-\frac{1}{3}}$ may not be regular.
(Solution): Consider $D = a^*bc^*$. Then $D_{\frac{1}{3}-\frac{1}{3}} \cap a^*b^* = \{a^n c^n \mid n \geq 1\}$, which is not regular.

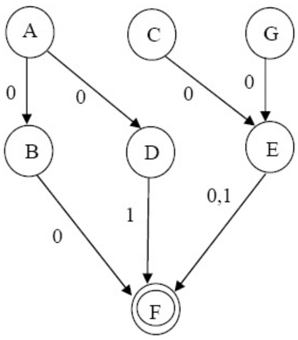


Figure 1: A finite automaton.

5. (20 pts) Consider a (deterministic or nondeterministic) finite automaton $M = (Q, \Sigma, \delta, q_0, F)$. Two states p, q are said to be equivalent (written as $p \approx_M q$) if

$$\forall z \in \Sigma^*, \delta^*(p, z) \cap F \neq \emptyset \text{ iff } \delta^*(q, z) \cap F \neq \emptyset.$$

Note that $\delta^*(s, w)$ denotes the set of states that can be reached from state s by reading input string w.

(a) (5 pts) With respect to M in Figure 1, what are the equivalence classes induced by $\approx_M$. (Note that each equivalence class is a subset of states.) Why?
(Solution)
$\{A, C, G\}, \{B\}, \{D\}, \{E\}, \{F\}$;
Note that $L(A) = L(C) = L(G) = \{00, 01\}$; $L(B) = \{0\}$; $L(D) = \{1\}$; $L(E) = \{0, 1\}$; $L(F) = \{\epsilon\}$, **where** $L(X)$ **is the set of words leading to acceptance from node** X.

(b) (5 pts) With respect to M in Figure 1, can you find two states p, q with $p \approx_M q$, but $\exists a \in \Sigma$, such that $\forall\, p' \in \delta(p, a)$ and $\forall\, q' \in \delta(q, a)$, $p' \not\approx_M q'$? Why?
(Solution) $A \approx_M C$ **although no two of** B, D, E **are equivalent.**

(c) (5 pts) Suppose we are given a DFA M'. Can you still find two states $p \approx_{M'} q$, but $\exists a \in \Sigma$, such that $\forall\, p' \in \delta(p, a)$ and $\forall\, q' \in \delta(q, a)$, $p' \not\approx_M q'$? Why?
(Solution) No.
$p' \not\approx_M q'$ **implies** $\exists z \in \Sigma^*$, $(\delta(p', z) \in F$ **and** $\delta(q', z) \notin F)$ **or** $(\delta(p', z) \notin F$ **and** $\delta(q', z) \in F)$. **Then for all** p, q **with** $p \xrightarrow{a} p'$ **and** $q \xrightarrow{a} q'$, **we have** $(\delta(p, az) \in F$ **and** $\delta(q, az) \notin F)$ **or** $(\delta(p, az) \notin F$ **and** $\delta(q, az) \in F)$. **Hence,** $p \not\approx_{M'} q$

(d) (5 pts) Consider the following notion of equivalence called *bisimulation equivalence*: With respect to FA M, two state p, q are *bisimilar*, written as $p \approx_M^b q$ iff the following three conditions are satisfied:
(1) $p \approx_M q$,
(2) For all $a \in \Sigma$ and all $p' \in \delta(p, a)$, there exists a state $q' \in \delta(q, a)$ such that $p' \approx_M^b q'$, and
(3) For all $a \in \Sigma$ and all $q' \in \delta(q, a)$, there exists a state $p' \in \delta(p, a)$ such that $p' \approx_M^b q'$.
(Note that the above involves a recursive definition.) With respect to the FA M given in Figure 1, find the equivalence class induced by $\approx_M^b$. Why?
(Solution) $\{A\}$, $\{C, G\}, \{B\}, \{D\}, \{E\}, \{F\}$; **because** $C \xrightarrow{0} E$, **but regardless of considering** $A \xrightarrow{0} B$ **or** $A \xrightarrow{0} D$, $E \not\approx_M^b B$ **and** $E \not\approx_M^b D$

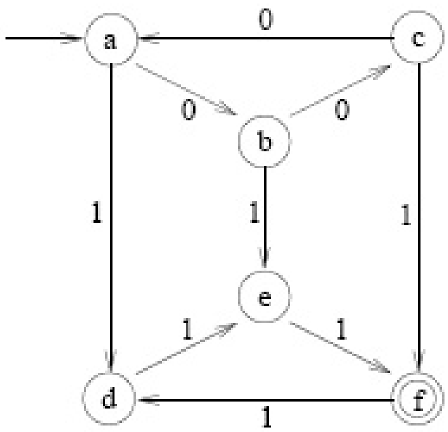


Figure 2: A DFA.

6. (10 pts) Consider the DFA shown in Figure 2. Use the state elimination method in the order e, b, d, c, f, a to find an equivalent regular expression. Show your derivation in detail by drawing all the intermediate automata (for which transitions are labeled with regular expressions).
(Solution): $(000)^*(001 \cup 011 \cup 111)(111)^*$.