Theory of Computation

Spring 2025, Homework # 4

Due: May 27, 2025

1. (20 pts) Give an example of an undecidable (i.e., non-recursive) subset of 1^*. Prove that your answer is correct. (Hint: Consider the list of all TMs $M_1, M_2, ..., M_k, ...$ Construct a language $L \subseteq 1^*$, and use the diagonalization method to show L to be non-recursive.)
   Sol: Consider $D = \{1^k \mid 1^k \notin L(M_k)\}$, where M_k is the k-th TM, as the set of TMs is countably infinite. The rest is by a standard diagonalization.

2. (20 pts) Is the language $L = \{\langle M, w\rangle \mid M$ at some point in time moves left while computing $w\}$ recursive? Why? Give a convincing argument.

   Sol: Yes. It is recursive. The following procedure can decide L:
   Simulate M on w, one of the following must occur:

   (a) During the first $|w|$ steps, M makes a left move. Accept, in this case; otherwise,

   (b) M moves into the region of the tape with a sequence of blank symbols. Either M makes a left move in $|Q|$ steps, or a state repeats. In the former case, accept; in the latter case, reject as M will move to the right forever.

   In the above, M is assumed to be a DTM. The NTM case is similar using a breadth-first simulation.

3. (20 pts) Let $L = \{\langle M_1, M_2\rangle \mid M_1, M_2$ are TMs such that for some input x, both M_1 and M_2 halt on $x\}$. Prove that L is r.e. but not recursive.
   Sol: We write s_i to denote the i^{th} string of Σ^* in alphabetical order. Consider the following algorithm:

   for ($i = 1$ to ∞)
   for ($j = 1$ to i)
     Simulate M_1 on s_j for i steps.
     Simulate M_2 on s_j for i steps.
     Accept if both M_1 and M_2 halted.

   It accepts L, so L is *r.e.*.

   To show that L is not recursive, it suffices to show that HP $\leq_m L$. We write K_x to denote a TM that halts if only if the input is x. The map $\langle M, x\rangle \rightarrow \langle K_x, M\rangle$ witnesses the reduction.

4. (20 pts) Let $T = \{\langle M\rangle \mid M$ is a TM that accepts w^R whenever it accepts $w\}$. Show that T is not decidable. Do not use Rice's theorem. (Here w^R is the reversal of w.)
   Sol: Reduce $HALT_{TM}$ c to T.

   For each $\langle M, w\rangle \in HALT_{TM}$, we define a TM M_w as follows: For each $w' \in \Sigma^*$,

   - reject if $w' \neq 01$ *or* 10;
   - accept if $w = 01$; and
   - simulate M on w if $w = 10$.
     - accept if M accepts w;
     - reject if M rejects w.

   Hence, we have $\langle M_w\rangle \in T \leftrightarrow L(M_w) = \{01, 10\} \leftrightarrow M$ accepts w. Since halting problem is undecidable, T is not decidable.

5. (20 pts) Suppose there are four languages A, B, C, and D. Each of the languages may or may not be recursively enumerable. However, we know the following about them: $A \leq_m B$, $B \leq_m C$, and $D \leq_m C$. Below are four statements. Indicate whether each one is

   (a) CERTAIN to be true, regardless of what languages A through D are.

   (b) MAYBE true, depending on what A through D are.

   (c) NEVER true, regardless of what A through D are.

   Justify your answers.

   (1) A is recursively enumerable but not recursive, and C is recursive.

   (2) A is not recursive, and D is not recursively enumerable.

   (3) If C is recursive, then the complement of D is recursive.

   (4) If C is recursively enumerable, then $B \cap D$ is recursively enumerable.

   Sol:

   (a) NEVER. If A is not recursive, C is not recursive since $A \leq_m C$.

   (b) MAYBE. Both $A = B = C = D = \{0^n 1^n | n \geq 0\}$ or $A = B = C = D = \neg\text{HP}$ are consistent with the relation $\leq_m$.

   (c) CERTAIN. $D \leq_m C$ implies that if C is recursive, then $\rightarrow D$ is recursive. D is recursive if and only if $\bar{D}$ is recursive.

   (d) CERTAIN. $B \leq_m C$, $D \leq_m C$ implies that if C is r.e., then both B, D are r.e.. If B, D are r.e., then $B \cap D$ must also be r.e..