Theory of Computation

Spring 2025, Homework # 3

Due: May 13, 2025

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1. (30 pts) A CFG $G = (V, \Sigma, P, S)$ (where V and Σ are the sets of nonterminals and terminals, resp.) is called *linear* if all productions are of the form
$A \to xBy, x, y \in \Sigma^*, B \in V$, or $A \to z, z \in \Sigma^*$.

    - Prove the following pumping lemma for linear languages:
    Let L be an infinite linear language. There exists some positive m such that $w \in L$, with $|w| \geq m$, w can be decomposed as $w = uvxyz$ with $|uvyz| \leq m$, $|vy| \geq 1$, and $uv^ixy^iz \in L, i = 0, 1, 2, ...$
    - Show that the language $L = \{w \in \{a, b, c\}^* \mid \#_a(w) + \#_b(w) = \#_c(w)\}$ is not linear, where $\#_a(w), \#_b(w), \#_c(w)$ denote the numbers of occurrences of a, b, c in w.

2. (20 pts) Show that the language $\{a^ib^j \mid i = k * j \text{ for some positive integer } k\}$ is not context-free. (Hint: Consider, for example, $a^{p^2}b^p$, where p is prime.)

3. (30 pts) Shuffle for CFLs

    - Give a counterexample to show that if L_1 and L_2 are both CFL, then $shuffle(L_1, L_2)$ need not be a CFL.
    - Prove that if L is CF and R is regular, then $shuffle(L, R)$ is a CFL.
    (Hint: Design a PDA to accept $shuffle(L, R)$.)

4. (20 pts) Let A and B be two disjoint languages over alphabet Σ. We say that language C separates A and B if $A \subseteq C$ and $B \subseteq \overline{C}$ (i.e., $B \subseteq (\Sigma^* - C)$). Show that any two disjoint *co-r.e.* languages are separable by some recursive language.
(Hint: Let A, B be two co-r.e. languages. Let TMs $M_{\overline{A}}$ and $M_{\overline{B}}$ accept $\overline{A}$ and $\overline{B}$, respectively.)