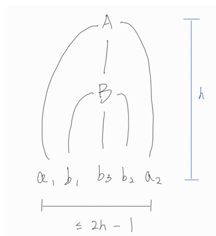
Theory of Computation

Spring 2025, Homework # 3

Due: May 13, 2025

1. Answer.

   - The proof process is similar to the Pumping Lemma for CFLs.
     The parse tree must in the form:

     Choose $m = 2|V| + 3$, let $w \in G$ with $|w| \geq m$ and τ the smallest parse tree for w. Let R be the first nonterminal that appears twice on the backbone. The root of the subtree generating vxy corresponds to the 1st occurrence of R. The root of the subtree generating x corresponds to the 2nd occurrence of R.
     We have $|vy| \geq 1$ (see Pumping Lemma for CFLs) and there are at most $|V|+1$ nonterminals from A to the 2nd R, so $|uvyz| \leq 2(|V|+1) \leq m$
   - Suppose L is linear. Let m be the pumping length of L and set $w = a^m c^{2m} b^m$, then no c contained in $uvyz$ which means $uv^2xy^2z \notin L$. Contradiction.

2. Answer.
   Call the language L. Suppose L is linear. Let p be a prime larger than the pumping length and set $w = a^{p^2} b^p$.

   - Case 1: vxy contains a and b
     Then we have $uv^ixy^iz \in L$ for all i. Let n_a (resp.n_b) denote the number of a (resp.b) in vy. When $i = 0$, $k = p$.
     k is either decrease or increase for large k, and $k \to \dfrac{n_a}{n_b}$ as $i \to \infty$. While integers between p and $\dfrac{n_a}{n_b}$ is finite. Contradiction.
   - Case 2: vxy contain only a (or b)
     $uv^2xy^2z \notin L$ since $|vxy| \leq p$ Contradiction.

3. Answer.

   - First, prove that if L is CF and R is regular, then shuffle(L, R) is a CFL. $P = (\{q_p\}, \Sigma, \Gamma, \delta_p, q_p, F_p)$. is a PDA accept L, $D = (Q_d, \Sigma, \delta_d, q_0, F_d)$ is a DFA accept R. Define δ :

     $$\delta(q, a, \alpha) = \begin{cases} (q, \beta) & \forall q \in Q_d \quad \text{if } \delta_p(q_p, a, \alpha) = (q_p, \beta) \\ (q', \alpha) & \qquad\qquad\quad \text{if } \delta_d(q, a) = q' \\ (q_f, \alpha) & \forall q \in F_d \quad \text{if } a = \varepsilon, \alpha = \varepsilon \end{cases} \tag{1}$$

     Then PDF $P' = (Q_d \cup \{q_f\}, \Sigma, \Gamma, \delta, q_0, \{q_f\})$ accept shuffle(R,L).
   - Let $L_1 = \{a^n b^n | n \geq 1\}$, $L_2 = \{c^n d^n | n \geq 1\}$, $R = a^* c^* b^* d^*$. Now shuffle$(L_1, L_2) \cap R$ is not a CFL.

4. Answer.
   Using the hint, and construct Turing machine M :

$$
\begin{aligned}
M = \text{"Run input x on } M_{\overline{A}} \text{ and } M_{\overline{B}} \text{ parallelly.} && (2)\\
\text{if } M_{\overline{A}} \text{ accept, reject} && (3)\\
\text{if } M_{\overline{B}} \text{ accept, accept"} && (4)
\end{aligned}
$$

   A and B are disjoint so $\overline{A} \cup \overline{B} = \Sigma^*$ and M halts on all input x.
   Claim $L(M)$ separates A and B.
   We know that $L(M) \subseteq \overline{B}$ and $\overline{L(M)} \subseteq \overline{A}$, which gives us $A \subseteq L(M)$ and $B \subseteq \overline{L(M)}$