Theory of Computation

Spring 2025, Homework # 2

Due: April 8, 2025

1. (15 pts) Let $L_1 \subseteq (a+b)^*$ be a set of strings. In each string in L_1, delete every b immediately preceding (i.e., before) an a to get the set L_2. For instance, if $L_1 = \{aabba, aa\}$, then $L_2 = \{aaba, aa\}$. You are asked to define two homomorphisms $h_1, h_2 : \{a, b, \hat{b}\}^* \to \{a, b\}^*)$, and write an expression for L_2 in terms of $h_1, h_2, h_1^{-1}, h_2^{-1}, R, L_1$, for some regular expression R. Explain why your answer is correct.

2. (15 pts) For a language L, let $Init(L) = \{x \mid xy \in L, \text{ for some } y \in \Sigma^*\}$. Let r, s, r_I and s_I be regular expressions for the languages R, S, $Init(R)$, and $Init(S)$, respectively. Using only these regular expressions and the operations $+$, concatenation, and $*$, give expressions for the following languages. Briefly explain why your answers work.

   (a) $Init(R \cup S)$

   (b) $Init(RS)$

   (c) $Init(R^*)$

3. (30 pts) Given a language L, we define (as discussed in class) the relation $\equiv_L$ for strings x and y in Σ^* as

$$x \equiv_L y \Leftrightarrow (\forall z \in \Sigma^*, xz \in L \Leftrightarrow yz \in L).$$

   Suppose we consider the following relations $\equiv^L$ and $\overset{L}{\equiv}$ instead

$$x \equiv^L y \Leftrightarrow (\forall z \in \Sigma^*, zx \in L \Leftrightarrow zy \in L).$$

$$x \overset{L}{\equiv} y \Leftrightarrow (\forall u, v \in \Sigma^*, uxv \in L \Leftrightarrow uyv \in L).$$

   Given a string w, let w^R be its reverse, i.e., the word obtained when reading w from right to left, e.g., $(abb)^R = bba$. For a language L, we let $L^R = \{w^R \mid w \in L\}$.

   (a) (5 pts) Prove in detail that $(x \equiv_L y) \Leftrightarrow (x^R \equiv^{L^R} y^R)$.

   (b) (10 pts) Consider langauge $L = (ab + ba)^*$. Determine the equivalence classes of the language $(ab + ba)^*$ under $\equiv_L$. Write down each of the equivalence classes using a regular expression.

   (c) (10 pts) Use the above equivalence classes to construct (draw) a DFA accepting $(ab + ba)^*$.

   (d) (5 pts) For $L = (ab + ba)^*$, can you find two strings x and y such that both $x \equiv_L y$ and $x \equiv^L y$ hold, but $x \overset{L}{\equiv} y$ does not hold. Justify your answer.

4. (20 pts) Construct a DFA for $\Sigma^* 1 (\Sigma\Sigma^*)^* 1 \Sigma^*$ with the smallest possible number of states, where $\Sigma = \{0, 1\}$. Prove that your DFA is the smallest possible. (Hint: To prove the DFA to be minimum, you may use the Myhill-Nerode theorem.)

5. (10 pts) Let B and D be two languages. We write $B \propto D$ if $B \subseteq D$ and D contains infinitely many strings that are not in B. Show that, if B and D are two regular languages where $B \propto D$, then we can find a regular language C such where $B \propto C \propto D$.

6. (10 pts) Give a right-linear grammar for the following regular language: $(00 \cup 1)^*$. Show your work in sufficient detail.